Research Article

An Extended Analytical Approach to Evaluating Monotonic Functions of Fuzzy Numbers

Arthur Seibel and Josef Schlattmann

Workgroup on System Technologies and Engineering Design Methodology, Hamburg University of Technology, 21073 Hamburg, Germany

Correspondence should be addressed to Arthur Seibel; arthur.seibel@tuhh.de

Received 18 September 2013; Accepted 26 December 2013; Published 11 February 2014

Academic Editor: Ning Xiong

Copyright © 2014 A. Seibel and J. Schlattmann. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper presents an extended analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers. The approach is based on a parametric $\alpha$-cut representation of fuzzy numbers and allows for the inclusion of parameter uncertainties into mathematical models.

1. Introduction

There is an increasing effort in the scientific community to provide suitable methods for the inclusion of uncertainties into mathematical models. One way to do so is to introduce parametric uncertainty by representing the uncertain model parameters as fuzzy numbers [1] and evaluating the model equations by means of Zadeh’s extension principle [2]. The evaluation of this classical formulation of the extension principle, however, turns out to be a highly complex task [3]. Fortunately, Buckley and Qu [4] provide an alternative formulation that operates on $\alpha$-cuts and is applicable to continuous functions of independent fuzzy numbers. Powerful numerical techniques have been developed to implement this alternative formulation [5]. These techniques are particularly suitable for very complex simulation models [6]. In engineering design [7], however, the mathematical equations are usually less complex, and hence analytical methods might be more suitable for the inclusion of parameter uncertainties into the computations. For this purpose, a practical analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers was introduced by the authors [8], which is based on the alternative formulation of the extension principle. In this paper, we extend this approach in terms of computational efficiency depending on certain monotonicity conditions.

An outline of this paper is as follows. In Section 2, we give a definition of fuzzy numbers and present two important types. In Section 3, we introduce the notion of a linguistic variable. In Section 4, we briefly recall Zadeh’s extension principle and introduce the alternative formulation based on $\alpha$-cuts. In Section 5, we describe our extended analytical approach and give four illustrative examples. In Section 6, a practical engineering application is presented. Finally, in Section 7, some conclusions are drawn.

2. Fuzzy Numbers

Fuzzy numbers are a special class of fuzzy sets [9], which can be defined as follows [1].

A normal, convex fuzzy set $\tilde{x}$ over the real line $\mathbb{R}$ is called fuzzy number if there is exactly one $x \in \mathbb{R}$ with $\mu_\tilde{x}(x) = 1$ and the membership function is at least piecewise continuous. The value $x$ is called the modal or peak value of $\tilde{x}$.

It is important to note that some authors consider normal, convex fuzzy sets with a core interval also as fuzzy numbers [10]. In [3, 6], these types of fuzzy numbers are denoted as fuzzy intervals. Furthermore, some authors define a fuzzy number having a compact support [11]. Although all concepts presented in this paper can be extended to these definitions of fuzzy numbers, we stick to the definition from [1].

Theoretically, an infinite number of possible types of fuzzy numbers can be defined. However, only few of them are important for engineering applications [6]. These typical fuzzy numbers shall be described in the following.
2.1. Triangular Fuzzy Numbers. Due to its very simple, linear membership function, the triangular fuzzy number (TFN) is the most frequently used fuzzy number in engineering. In order to define a TFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} 1 + \frac{x - \bar{x}}{\tau^L}, & \bar{x} - \tau^L \leq x \leq \bar{x}, \\ 1 - \frac{x - \bar{x}}{\tau^R}, & \bar{x} < x \leq \bar{x} + \tau^R, \end{cases}$$

we use the parametric notation [6]

$$\tilde{x} = \text{tfn}(\bar{x}, \tau^L, \tau^R),$$

where \(\bar{x}\) denotes the modal value, \(\tau^L\) denotes the left-hand, and \(\tau^R\) denotes the right-hand spread of \(\tilde{x}\) (cf. Figure 1). If \(\tau^L = \tau^R\), the TFN is called symmetric. Its \(\alpha\)-cuts \(x(\alpha) = [x^L(\alpha), x^R(\alpha)]\) result from the inverse functions of (1) with respect to \(x\):

$$x^L(\alpha) = \bar{x} - \tau^L \sqrt{2 \ln(\alpha)}, \quad 0 < \alpha \leq 1,$$

$$x^R(\alpha) = \bar{x} + \tau^R \sqrt{2 \ln(\alpha)}, \quad 0 < \alpha \leq 1.$$ 

2.2. Gaussian Fuzzy Numbers. Another widely used fuzzy number in engineering is the Gaussian fuzzy number (GFN), which is based on the normal distribution from probability theory. In order to define such a GFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} \exp \left[ -\frac{1}{2} \left( \frac{x - \bar{x}}{\sigma^L} \right)^2 \right], & x \leq \bar{x}, \\ \exp \left[ -\frac{1}{2} \left( \frac{x - \bar{x}}{\sigma^R} \right)^2 \right], & x > \bar{x}, \end{cases}$$

we use the parametric notation [6]

$$\tilde{x} = \text{gfn}(\bar{x}, \sigma^L, \sigma^R),$$

where \(\bar{x}\) denotes the modal value, \(\sigma^L\) denotes the left-hand, and \(\sigma^R\) denotes the right-hand standard deviation of \(\tilde{x}\) (cf. Figure 2). If \(\sigma^L = \sigma^R\), the GFN is called symmetric. Its \(\alpha\)-cuts \(x(\alpha) = [x^L(\alpha), x^R(\alpha)]\) result in

$$x^L(\alpha) = \bar{x} - \sigma^L \sqrt{-2 \ln(\alpha)}, \quad 0 < \alpha \leq 1,$$

$$x^R(\alpha) = \bar{x} + \sigma^R \sqrt{-2 \ln(\alpha)}, \quad 0 < \alpha \leq 1.$$ 

3. Linguistic Variables

In decision analysis, linguistic variables are of particular importance [12]. A linguistic variable \(V_L\) is a collection of subsets containing the following elements:

(i) \(G\): set of syntactic rules (e.g., in terms of a grammar) for the linguistic quantification of \(V_L\);
(ii) \(T\): set of terms \(t_i, i \in \mathbb{N}\), resulting from \(G\);
(iii) \(S\): set of semantic rules that assign every term \(t_i\) to its (physical) meaning in terms of a fuzzy number \(\tilde{t}_i\);
(iv) \(X\): (physically relevant) universal set with the (crisp) elements \(x\).

Figure 3 illustrates a possible description of the linguistic variable color. It is based on the continuous spectrum of the wave length \(\lambda\) of visible light: \(X = \{\lambda \in \mathbb{R} \mid 380 \leq \lambda \leq 780\}\) nm. By subjective color perception, the colors \(t_i\) are chosen from the set \(T = \{\text{violet, blue, cyan, green, yellow, red}\}\) of possible colors. Each term \(t_i \in T\) is represented as a fuzzy number \(\tilde{t}_i\) over the universal set \(X\).

For an easier handling with linguistic variables, they can be transformed into the unit interval \([0,1]\). These types of linguistic variables are referred to as normalized linguistic variables [12].

4. Extension Principle

Zadeh's extension principle [2] allows for extending any real-valued function to a function of fuzzy numbers. More specifically, let \(\tilde{x}_1, \ldots, \tilde{x}_n\) be \(n\) independent or noninteractive fuzzy numbers, and let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a function with
5. Extended Analytical Approach

Basically, our extended analytical approach can be classified into three parts depending on the monotonicity of $f$: a reduced [8], a general [8], and an extended part.

5.1. Reduced Part. Let the continuous function $f$ be (strictly) monotonic increasing in $x_i$, $i = 1, \ldots, k$, and (strictly) monotonic decreasing in $x_j$, $j = 1, \ldots, \ell$, in the domain of interest, and let $k + \ell = n$. Then, the minimum values of $f$ inside of every subdomain $\Omega(\alpha)$ are always found at the left boundaries of $x_j(\alpha)$ and the right boundaries of $x_i(\alpha)$ and its maximum values at the right boundaries of $x_j(\alpha)$ and the left boundaries of $x_i(\alpha)$, respectively. In such case, the $\alpha$-cuts $y(\alpha) = [y^L(\alpha), y^R(\alpha)]$ of $\tilde{y}$ become

\begin{align}
y^L(\alpha) &= f\left(x_i^L(\alpha), x_j^R(\alpha)\right), \quad 0 < \alpha \leq 1,
y^R(\alpha) &= f\left(x_i^R(\alpha), x_j^L(\alpha)\right), \quad 0 < \alpha \leq 1,
\end{align}

with $x_m(\alpha) = [x_m^L(\alpha), x_m^R(\alpha)], m = 1, \ldots, n$. If (9) is invertible with respect to $\alpha$, then the membership function of $\tilde{y}$ yields

\begin{align}
\mu_{\tilde{y}}(y) = \begin{cases} y^L(\alpha)^{-1}, & y < y^L(\alpha), \\
y^R(\alpha)^{-1}, & y^L(\alpha) < y < y^R(\alpha), \\
y^R(\alpha), & y^R(\alpha) \leq y \leq y^L(\alpha), & 0 < y \leq y^L(\alpha). 
\end{cases}
\end{align}

This reduced part of our approach can be viewed as an analytical version of the short transformation method [14]. Basically, it is equivalent to Lemma 3 from [15] or Corollary 2 from [16].

Example 1. The function $f_1 : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ with

\begin{align}
y_1 &= f_1(x_1, x_2) = \frac{x_1}{x_1 + x_2}
\end{align}

shall be evaluated for the two fuzzy numbers $\tilde{x}_1 = \text{tnf}(2, 2, 3)$ and $\tilde{x}_2 = \text{tnf}(2, 2, 2)$. Since

\begin{align}
\frac{\partial f_1}{\partial x_1} &= \frac{x_2}{(x_1 + x_2)^2} > 0, \\
\frac{\partial f_1}{\partial x_2} &= \frac{-x_1}{(x_1 + x_2)^2} < 0,
\end{align}

the function $f_1$ is (strictly) monotonic increasing in $x_1$ and (strictly) monotonic decreasing in $x_2$ in the domain $\text{supp}(\tilde{x}_1) \times \text{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)$. Hence, the $\alpha$-cuts $y_1(\alpha) = [y_1^L(\alpha), y_1^R(\alpha)]$ of $\tilde{y}_1$ are

\begin{align}
y_1^L(\alpha) &= f_1\left(x_1^L(\alpha), x_2^R(\alpha)\right) = \frac{1}{2} \alpha, \\
y_1^R(\alpha) &= f_1\left(x_1^L(\alpha), x_2^L(\alpha)\right) = \frac{3\alpha - 5}{\alpha - 5}.
\end{align}

With $y_1^L(0) = 0$, $y_1^L(1) = 0.5 = y_1^R(1)$, and $y_1^R(0) = 1$, the membership function of $\tilde{y}_1$ yields

\begin{align}
\mu_{\tilde{y}_1}(y) = \begin{cases} 2y, & 0 < y \leq 0.5, \\
5(y - 1), & 0.5 < y < 1.
\end{cases}
\end{align}

5.2. General Part. Unfortunately, the reduced part of our approach is only valid if the function $f$ does not change its monotonicity within the domain of interest. However, we know from [17, 18] that the global extrema of any monotonic function $f$ are always found at the corner points of $\Omega(\alpha)$. Hence, in order to obtain the analytical solution, we can always proceed as follows.
(1) Evaluate the function \( f \) for all the \( 2^n \) permutations of the interval boundaries of \( x_m^{l}(\alpha), x_m^{r}(\alpha) \), \( m = 1, \ldots, n \). For example, if \( n = 2 \), then compute
\[
\begin{align*}
y_{2}\text{LL}(\alpha) &= f \left( x_{1}^{l}(\alpha), x_{2}^{l}(\alpha) \right), \\
y_{2}\text{LR}(\alpha) &= f \left( x_{1}^{l}(\alpha), x_{2}^{r}(\alpha) \right), \\
y_{2}\text{RL}(\alpha) &= f \left( x_{1}^{r}(\alpha), x_{2}^{l}(\alpha) \right), \\
y_{2}\text{RR}(\alpha) &= f \left( x_{1}^{r}(\alpha), x_{2}^{r}(\alpha) \right). \quad \text{(15)}
\end{align*}
\]

(2) Plot these solution candidates in the same diagram.

(3) The analytical solution then corresponds to the maximum envelope formed by the possible solution candidates.

This general part of our approach can be viewed as an analytical version of the reduced transformation method \cite{19}. Basically, it is equivalent to Lemma 2 from \cite{15} or Corollary 1 from \cite{16}.

**Example 2.** Next, the function \( f_2 : \mathbb{R}^2 \rightarrow \mathbb{R} \) with
\[
y_2 = f_2(x_1, x_2) = \frac{(x_1 - 0.2)(x_2 - 1)}{x_1 + x_2} \quad \text{(16)}
\]
shall be evaluated for the two fuzzy numbers from Example 1.

Since
\[
\begin{align*}
\frac{\partial f_2}{\partial x_1} &= \frac{(x_2 - 1)(x_2 + 0.2)}{(x_1 + x_2)^2} \quad \text{for} \quad 0 < x_1 \leq 1, \\
\frac{\partial f_2}{\partial x_2} &= \frac{(x_1 - 0.2)(x_1 + 1)}{(x_1 + x_2)^2} \quad \text{for} \quad 0 < x_2 < 1, \quad \text{(17)}
\end{align*}
\]
the function \( f_2 \) changes its monotonicity within the domain \( \text{supp}(x_1) \times \text{supp}(x_2) = (0, 5) \times (0, 4) \). Hence, the general part of our approach should be applied. The solution candidates for \( y_2(\alpha) \) are
\[
\begin{align*}
y_{2}\text{LL}(\alpha) &= f_2 \left( x_{1}^{l}(\alpha), x_{2}^{l}(\alpha) \right) = \frac{(10\alpha - 1)(2\alpha - 1)}{20\alpha}, \\
y_{2}\text{LR}(\alpha) &= f_2 \left( x_{1}^{l}(\alpha), x_{2}^{r}(\alpha) \right) = \frac{(10\alpha - 1)(3 - 2\alpha)}{20\alpha}, \\
y_{2}\text{RL}(\alpha) &= f_2 \left( x_{1}^{r}(\alpha), x_{2}^{l}(\alpha) \right) = \frac{3(5\alpha - 8)(2\alpha - 1)}{5\alpha - 9}, \\
y_{2}\text{RR}(\alpha) &= f_2 \left( x_{1}^{r}(\alpha), x_{2}^{r}(\alpha) \right) = \frac{3(5\alpha - 8)(3 - 2\alpha)}{5\alpha - 9}. \quad \text{(18)}
\end{align*}
\]

We can see from their plots in Figure 4 that the left branch of the maximum envelope, illustrated by the gray area, is formed by \( y_{2}\text{RR} \) for \( 0 < \alpha < 0.5 \) and by \( y_{2}\text{LL} \) for \( 0.5 < \alpha < 1 \), where the value 0.5 corresponds to the intersection point between \( y_{2}\text{RL} \) and \( y_{2}\text{RR} \). Its right branch, on the other hand, is formed by \( y_{2}\text{LL} \) for \( 0 < \alpha < 0.02 \) and by \( y_{2}\text{RR} \) for \( 0.02 < \alpha < 1 \), where the value 0.02 corresponds to the intersection point between \( y_{2}\text{LL} \) and \( y_{2}\text{RR} \). Note that the value 0.02 is only approximate. Hence, the \( \alpha \)-cuts \( y_2(\alpha) = [y_2^1(\alpha), y_2^2(\alpha)] \) of \( y_2 \) are
\[
\begin{align*}
y_{2}\text{L}(\alpha) &= \left\{ \begin{array}{ll}
3(5\alpha - 8)(2\alpha - 1), & 0 < \alpha \leq 0.5, \\
5\alpha - 5, & 0.5 < \alpha \leq 1,
\end{array} \right. \\
y_{2}\text{R}(\alpha) &= \left\{ \begin{array}{ll}
(10\alpha - 1)(2\alpha - 1), & 0 < \alpha < 0.02, \\
20\alpha, & 0.02 < \alpha < 0.5, \\
3(5\alpha - 8)(3 - 2\alpha), & 0.5 < \alpha \leq 1.
\end{array} \right. \quad \text{(19)}
\end{align*}
\]

With \( y_{2}\text{L}(0) = -0.96, y_{2}\text{L}(0.5) = 0, y_{2}\text{L}(1) = 0.45 = y_{2}\text{R}(1), y_{2}\text{R}(0.02) \approx 1.57 \), and \( \lim_{\alpha \rightarrow 0} y_{2}\text{R}(\alpha) = \infty \), the membership function of \( y_2 \) yields
\[
\begin{align*}
\mu_{y_2}(y) &= \left\{ \begin{array}{ll}
\frac{21}{20} + \frac{1}{12}y - \frac{1}{60}\sqrt{A}, & -0.96 < y \leq 0, \\
\frac{3}{10} + \frac{1}{2}y + \frac{1}{10}\sqrt{B}, & 0 < y \leq 0.45, \\
\frac{31}{20} - \frac{5}{12}y - \frac{1}{60}\sqrt{C}, & 0.45 < y \leq 1.57, \\
\frac{3}{10} + \frac{1}{2}y - \frac{1}{10}\sqrt{D}, & 1.57 < y \leq \infty.
\end{array} \right. \quad \text{(20)}
\end{align*}
\]

![Figure 4: Solution candidates from Example 2.](image)
where

\[ A = 25y^2 - 2370y + 1089, \]
\[ B = 25y^2 + 30y + 4, \]
\[ C = 625y^2 + 750y + 9, \]
\[ D = 25y^2 + 30y + 4. \]

(21)

5.3. Extended Part. A drawback of the general part of our approach is the fact that a total of \(2^n\) function evaluations have to be carried out to compute the possible solution candidates. However, if some of the variables do not change their monotonicity within the domain of interest, that is, if \(k + \ell = q < n\), we can adapt our approach as follows.

(1) Evaluate the function \(f\) for \(x^L_1(\alpha), i = 1, \ldots, k\), and \(x^R_\ell(\alpha), j = 1, \ldots, \ell\), including all the \(2^{n-q}\) permutations of the interval boundaries of \(x_p(\alpha) = [x^L_p(\alpha), x^R_p(\alpha)], p = 1, \ldots, n - q\), to compute the solution candidates for \(y^L(\alpha)\).

(2) Evaluate the function \(f\) for \(x^R_i(\alpha), i = 1, \ldots, k\), and \(x^L_j(\alpha), j = 1, \ldots, \ell\), including all the \(2^{n-q}\) permutations of the interval boundaries of \(x_p(\alpha) = [x^L_p(\alpha), x^R_p(\alpha)], p = 1, \ldots, n - q\), to compute the solution candidates for \(y^R(\alpha)\).

(3) Plot these solution candidates in the same diagram.

(4) The analytical solution then corresponds to the maximum envelope formed by the possible solution candidates.

This extended part of our approach requires a total of \(2^{n-q+1}\) function evaluations. Note that, for \(q = 1\), the general and the extended part both lead to \(2^n\) function evaluations.

Example 3. Now, the function \(f_3 : \mathbb{R}^2_+ \rightarrow \mathbb{R}\) with

\[ y_3 = f_3(x_1, x_2) = \frac{x_1 - 0.2}{x_1 + x_2} \] (22)

shall be evaluated for the two fuzzy numbers from Example 1. Since

\[ \frac{\partial f_3}{\partial x_1} = \frac{x_2 + 0.2}{(x_1 + x_2)^2} > 0, \]
\[ \frac{\partial f_3}{\partial x_2} = \frac{0.2 - x_1}{(x_1 + x_2)^2} \begin{cases} \geq 0, & 0 < x_1 \leq 0.2, \\ < 0, & 0.2 < x_1 < 5, \end{cases} \] (23)

the function \(f_3\) is (strictly) monotonic increasing in \(x_1\) but changes its monotonicity in \(x_2\) within the domain \(\text{supp}(\tilde{x}_1) \times \text{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)\). Hence, the extended part of our approach should be applied. Note that here, \(q = 1\). The solution candidates for \(y^L_3(\alpha)\) are

\[ y^L_3(\alpha) = f_3(x^L_1(\alpha), x^L_2(\alpha)) = \frac{10\alpha - 1}{20\alpha}, \]
\[ y^R_3(\alpha) = f_3(x^R_1(\alpha), x^R_2(\alpha)) = \frac{1}{2} \frac{\alpha - 1}{20}, \] (24)

and for \(y^R_3(\alpha)\),

\[ y^L_3(\alpha) = f_3(x^L_1(\alpha), x^L_2(\alpha)) = \frac{3.5\alpha - 8}{5\alpha - 5}, \]
\[ y^R_3(\alpha) = f_3(x^R_1(\alpha), x^R_2(\alpha)) = \frac{3}{5} \frac{5\alpha - 8}{5\alpha - 9}. \] (25)

We can see from their plots in Figure 5 that the left branch of the maximum envelope is formed by \(y^L_3\) for \(0 < \alpha \leq 0.1\) and by \(y^R_3\) for \(0.1 < \alpha \leq 1\), where the value 0.1 corresponds to their intersection point. Its right branch, on the other hand, is entirely formed by \(y^R_3\). Hence, the \(\alpha\)-cuts \(y_3(\alpha) = [y^L_3(\alpha), y^R_3(\alpha)]\) of \(\tilde{y}_3\) are

\[ y^L_3(\alpha) = \begin{cases} \frac{10\alpha - 1}{20\alpha}, & 0 < \alpha \leq 0.1, \\ \frac{1}{2} \frac{\alpha - 1}{20}, & 0.1 < \alpha \leq 1, \end{cases} \]
\[ y^R_3(\alpha) = \begin{cases} \frac{3.5\alpha - 8}{5\alpha - 5}, & 0 < \alpha \leq 1. \end{cases} \] (26)
Finally, the function to their intersection point. Its right branch, on the other hand, is entirely formed by \( y_4^{RRL} \). Hence, the \( \alpha \)-cuts \( y_4^i(\alpha) = [y_4^{L}(\alpha), y_4^{R}(\alpha)] \) of \( \tilde{y}_4 \) are

\[
\begin{align*}
y_4^{L}(\alpha) &= \begin{cases} 4\alpha^2 - 1 & 0 < \alpha \leq 0.5, \\ 4\alpha^2 - 1 & 0.5 < \alpha \leq 1, \end{cases} \\
y_4^{R}(\alpha) &= \frac{6\alpha^2 - 22\alpha + 19}{9 - 2\alpha}, & 0 < \alpha \leq 1.
\end{align*}
\]

With \( \lim_{\alpha \to 0} y_4^{L}(\alpha) = -\infty, y_4^{L}(0.5) = 0, y_4^{L}(1) = 0.43 = y_4^{R}(4), \) and \( y_4^{R}(0) = 2.11, \) the membership function of \( \tilde{y}_4 \) yields

\[
\begin{align*}
\mu_{\tilde{y}_4}(y) &= \begin{cases} \frac{7}{8} y + \frac{1}{8} \sqrt{E}, & -\infty < y \leq 0, \\ \frac{1}{4} y + \frac{1}{4} \sqrt{F}, & 0 < y \leq 0.43, \\ \frac{11 - \frac{1}{6}}{6} y - \frac{1}{6} \sqrt{G}, & 0.43 < y < 2.11, \end{cases}
\end{align*}
\]

where

\[
\begin{align*}
E &= 49y^2 + 16, \\
F &= y^2 + 20y + 4, \\
G &= y^2 + 32y + 7.
\end{align*}
\]

### 6. Engineering Application

In order to illustrate the extended analytical approach in a more practical context, we consider a simplified version of
Table 1: Linguistic weights of the criteria and ratings of the alternative materials.

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Low weight</th>
<th>Low cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight $w_j$</td>
<td>H</td>
<td>MH</td>
</tr>
<tr>
<td>Alternative $i$</td>
<td>1 Polymer composite</td>
<td>H</td>
</tr>
<tr>
<td></td>
<td>2 Aluminum alloy</td>
<td>MH</td>
</tr>
</tbody>
</table>

the case study from [20], where the material for an automotive bumper beam has to be selected. Here, two alternative materials (polymer composite and aluminum alloy) have to be evaluated against the criteria low weight and low cost using the normalized linguistic variable value scale from Figure 7. The corresponding linguistic weights and ratings are summarized in Table 1.

For computing the fuzzy overall rating $\tilde{r}_i$ of each alternative $i$, we use the fuzzy weighted average

$$\tilde{r}_i = \frac{\sum_{j=1}^n w_j \tilde{r}_{ij}}{\sum_{j=1}^n w_j},$$

where $\tilde{r}_{ij}$ denotes the fuzzy rating of the alternative $i$ according the criterion $j$ and $\tilde{w}_j$ denotes the fuzzy weight of the criterion $j$ (see Table 1). Since

$$\frac{\partial r_i}{\partial w_k} = \frac{r_{ik}}{\sum_{j=1}^n w_j^2},$$

$$\frac{\partial r_i}{\partial r_{kj}} = \frac{w_j}{\sum_{j=1}^n w_j},$$

$r_i$ is (strictly) monotonic increasing in $r_{ij}$ but may change its monotonicity in $w_j$ within the domain $(0,1)^2$. Hence, the extended part of our approach should be applied. The solution candidates for $r_1^L(\alpha)$ are

$$r_1^{L1}(\alpha) = r_1(w^L_1(\alpha), w^L_2(\alpha), r^L_{11}(\alpha), r^L_{12}(\alpha))$$

$$= 1 \frac{2\alpha^2 + 11\alpha + 16}{6 - 2\alpha + 7},$$

$$r_1^{L2}(\alpha) = r_1(w^L_1(\alpha), w^L_2(\alpha), r^L_{11}(\alpha), r^L_{12}(\alpha))$$

$$= 1 \frac{13\alpha + 12}{54 - 9},$$

and for $r_1^R(\alpha)$,

$$r_1^{R1}(\alpha) = r_1(w^R_1(\alpha), w^R_2(\alpha), r^R_{11}(\alpha), r^R_{12}(\alpha))$$

$$= 1 \frac{2\alpha^2 - 9\alpha + 6}{6 - 2\alpha - 11}.$$
Furthermore, the solution candidates for \( r_2^L(\alpha) \) are

\[
\begin{align*}
  r_{2 \text{LLL}}^L(\alpha) &= r_2^L\left(w_1^L(\alpha), w_2^L(\alpha), r_{21}^L(\alpha), r_{22}^L(\alpha)\right) \\
  &= \frac{1}{3} (\alpha + 3)(\alpha + 2), \\
  r_{2 \text{LRL}}^L(\alpha) &= r_2^L\left(w_1^L(\alpha), w_2^R(\alpha), r_{21}^L(\alpha), r_{22}^L(\alpha)\right) \\
  &= \frac{2}{9} \alpha + \frac{2}{9}, \\
  r_{2 \text{RLL}}^L(\alpha) &= r_2^L\left(w_1^R(\alpha), w_2^L(\alpha), r_{21}^L(\alpha), r_{22}^L(\alpha)\right) \\
  &= \frac{1}{9} \alpha + \frac{1}{3}, \\
  r_{2 \text{RRR}}^L(\alpha) &= r_2^L\left(w_1^R(\alpha), w_2^R(\alpha), r_{21}^L(\alpha), r_{22}^L(\alpha)\right) \\
  &= \frac{1}{3} \alpha^2 - 4\alpha - 9,
\end{align*}
\]

and for \( r_2^R(\alpha) \),

\[
\begin{align*}
  r_{2 \text{LRR}}^R(\alpha) &= r_2^R\left(w_1^L(\alpha), w_2^R(\alpha), r_{21}^R(\alpha), r_{22}^R(\alpha)\right) \\
  &= \frac{1}{3} \alpha^2 - 13, \\
  r_{2 \text{RLL}}^R(\alpha) &= r_2^R\left(w_1^L(\alpha), w_2^L(\alpha), r_{21}^R(\alpha), r_{22}^R(\alpha)\right) \\
  &= -\frac{1}{9} \alpha + \frac{5}{9},
\end{align*}
\]

Thus, for \( r_2(\alpha) = [r_2^L(\alpha), r_2^R(\alpha)] \),

\[
\begin{align*}
  r_2^R(\alpha) &= r_2^R\left(w_1^R(\alpha), w_2^R(\alpha), r_{21}^R(\alpha), r_{22}^R(\alpha)\right) \\
  &= -\frac{2}{9} \alpha + \frac{2}{3}, \\
  r_2^R(\alpha) &= r_2^R\left(w_1^R(\alpha), w_2^R(\alpha), r_{21}^R(\alpha), r_{22}^R(\alpha)\right) \\
  &= -\frac{1}{3} \alpha - 5, \quad 0.22 < r_2 < 0.67.
\end{align*}
\]

The membership functions of \( \tilde{r}_1 \) and \( \tilde{r}_2 \) are illustrated in Figure 10. There, we can see that polymer composite seems to be more appropriate as bumper beam material than aluminum alloy.
7. Conclusions

The proposed extended analytical approach is a very practical tool for the inclusion of parameter uncertainties into mathematical models. It is valid for continuous, monotonic functions of independent fuzzy numbers but can also be applied to fuzzy intervals as defined in [3, 6].

An analytical solution has the advantage that the degrees of membership of the fuzzy output can be computed for any value within the support, whereas a numerical solution only provides a finite number of values. Furthermore, our approach also allows a symbolic processing of uncertainties.

In further research activities, this approach shall be generalized to nonmonotonic functions of independent fuzzy numbers, where the influence of interdependency shall be investigated as well.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This publication was supported by the German Research Foundation (DFG) and Hamburg University of Technology (TUHH) in the funding programme “Open Access Publishing.”

References


Submit your manuscripts at http://www.hindawi.com