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On Second Order Contributions to Ship Waves and Wave Resistance
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von

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To be presented at the 6th-Symposium
on Naval Hydrodynamics

Washington 1966
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The O.N.R. - N.S.F. Symposium on Wave Resistance Theory in Ann Arbor held 1963 made clear that current research is focussed around the following items:
(A) Determination of quantities from the wave pattern representative for wave resistance.
(B) Formal and semi-empirical corrections to the classical linearized theory.
(C) More refined techniques for optimizing ship forms within linear theory.

Within the present paper I shall report on work done since then which might provide material to enforce progress in any of these directions. The "pièce de résistance" of this contribution is the gradual evolution of a computer programm which in a rationalized way gives the basic information of flow and wave components due to typical singularities, - (as discrete doublets, doublet struts, continuous parabolic distributions on submerged lines, infinite and truncated vertical planes) - all within linearized approach. This information lends itself readily for application to item (A). Any method proposed for determination of energy flow from characteristics of the flow, in peculiar from the geometry of wave pattern, can be tested for accuracy and for consistency on such a theoretical wave field available numerically before entering into expensive experimental work which provides in general too little reliable information on optimal choice of region where to perform measurements. The overwhelming part of the methods proposed for (A) is implicitly based on validity of certain asymptotic representations for the wave pattern. Only numerical calculations can tell what distances are already large enough, especially regarding decay of the so called "local flow components" in order that such representations may be applied.
For (B), the theories of wave resistance used nowadays are of second order, based on linearized flow models. For calculation of wave resistance, only a far field component of this flow has to be known explicitly; for any consistent approach to third order resistance contributions, however, the knowledge of the entire first order flow is essential. Aside from some semi-empirical approaches to alternate formulations of linear theory, which we shall submit to some critical examen, and aside from indirect approach as successfully carried out by Kajitani recently, the tool for a systematic perturbation attack to the higher order flow components has been provided by Wehausen [1, 2, 3] in a series of papers starting with that read before this audience in 1956 up to his contribution to the Ann Arbor conference. As, however, the step to formulate resistance expressions was not performed, credit is generally given to Sisov [4] for first dealing with these. We should, nevertheless, be aware, that expressions given by Sisov so far essentially contain divergent integrals due to selection of improper radiation condition for Green's function of pressure point. In our present investigation, we will rederive some of Sisov's results from a Green's theorem approach essentially following Wehausen. We will, in particular, show up some simplifications which make calculations straightforward once a Fourier representation of first order flow components is given. It will become evident that integration over undisturbed free surface has to be performed only in a small domain where local flow is significant; third order wave resistance is, therefore, much more tractable to numerical evaluation than is apparent from what was formulated by Sisov, provided we decide on an appropriate definition of wave resistance.

We decided to deviate from Wehausen's approach by some simplifications regarding the actual flow boundaries. However, the resulting expressions found for third order resistance depend in a simple manner only on ship's offsets and on first order velocity components. We, therefore, feel that these deviations at least have not introduced artificial complications against results still to be found from more refined analysis.

Regarding the third problem, i.e. ships of minimum resistance within lowest order theory, our investigation should throw light on the question to what degree third order contributions might counteract the tendencies predicted. At the present stage, however, our calculations are limited to a two-parameter class of hull forms having parabolic waterlines. This is mainly due to the fact that we preferred analytical evaluation of integrals over the geometry of the ship. An extension of our program for local flow, to include contributions from
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empirical surface elements is feasible, but loss of closed integration would probably increase time for computation and weaken control of accuracy. Moreover, the necessary degree of hull-subdivision will in general depend on Froude number and is not known beforehand. Even for analytical ship forms, the development of formal expressions for closed integration cannot be done by the computer and provides many opportunities for errors in evaluation of singular regions of integrands for local flow.

Description of analysis to derive potential and wave resistance.

We shall essentially follow the approach of Wehausen [3], but modify it for flow in a tank of rectangular cross section. This will simplify the formulation of radiation conditions for the flow and allows the use of a Green's function in a Fourier series representation regarding the ordinate \( y \) chosen in direction perpendicular to the vertical tank walls. The ship's motion is in the \( +x \)-direction with speed \( c \), the \( z \) coordinate is taken vertically upwards to conform with earlier work [6]. As far as possible we otherwise use notation consistent with [3]. However, direction of normal vectors is reversed resulting from our definition of Green's function with an opposite sign. Extension of results to unrestricted water is straightforward.

1. Derivation of second order potential.

We introduce dimensionless coordinates as \( X = 2x/L \), \( Y = 2y/L \), \( Z = 2z/L \), where \( L \) is the ship's length. The velocity potential is nondimensionalized as \( \psi = 2\varphi/Lc \). As speed parameter we use \( \varepsilon = gL/2c^2 \).

Let \( Y^r \in \mathcal{F}(X,Z) \) be the dimensionless representation of the hull geometry, where \( B \) is the ship's breadth. \( \varepsilon = B/L \) will serve as a perturbation parameter and is considered as a small quantity. Let \( X = X_0 \) and \( X = X_e \) be the equations of two vertical control planes \( S_a \) and \( S_e \) ahead of and behind the ship. Let \( S_b \) stand for the tank bottom plane \( Z = -H \). Let \( Y = 1 \) be the equations of the vertical tank walls \( S_r \) and \( S_l \), where \( T = b/L \), \( b = \) tank width. Let \( S_f \) stand for the free surface \( Z = \bar{S}(X,Y) \) for \( X_e < X < X_0 \), \( T < Y < T \); let \( S_f^\circ \) stand for the undisturbed free surface \( Z = 0 \) with the waterplane area of the ship excluded. Let \( S_w \) stand for the wetted surface of the ship and \( S_w^\circ \) stand for the part of the surface up to \( Z = 0 \). Let \( D \) stand for the domain of the complete flow, bounded by \( S_w, S_f, S_a, S_e, S_r, S_l, \) and \( S_b \). Let \( D^\circ \) describe the corresponding domain if \( S_w \) and \( S_f \) are replaced by \( S_w^\circ \) and \( S_f^\circ \). Let \( \psi^\circ \) stand for the Michell type first approximation to the exact potential \( \psi \), let \( P \) stand for a point...
in D or D° with coordinates X, Y, Z and let P' represent a point on a boundary surface with coordinates ξ, η and ζ. Let G(P, P') stand for the potential of a source of output 4π as defined in the Appendix.

The functions ψ, ψ" and G of the variables X, Y, Z are subject to the following set of conditions:

A. Laplace equation: \( \Delta \psi = 0 \) in D, \( \Delta \psi^{(ii)} = 0 \) in \( D° \),

where \( \Delta \) stands for \( \delta^2 / \delta x^2 + \delta^2 / \delta y^2 + \delta^2 / \delta z^2 \) and \( \delta \) means the Dirac delta function, which is zero if P is unequal P'; (condition A. implies that G becomes singular as \( -1 / |P-P'| \)).

B. On \( S_f° \) we have (linearized free surface condition):

\[ \gamma_0 \psi_n^{(ii)} + \psi_{xx} = 0; \quad \gamma_0 G_x + G_{xx} = 0. \]

For the exact potential \( \psi \), no such conditions hold. But we define a function \( \delta (X, Y) \) by

\[ \gamma_0 \psi_z + \psi_{xx} = \delta (X, Y). \]

C. On \( S_f \) and \( S_r \) we have: \( \psi_y = 0; \quad \psi_{yy} = 0; \quad G_y = 0. \)

D. On \( S_b \) we have: \( \psi_z = 0; \quad \psi_{zz} = \sigma; \quad G_z = 0. \)

E. On \( S_w \) we have: \( \psi_n = \pm \varepsilon F_x / \sqrt{\varepsilon_x^2 F_x^2 + \varepsilon_z^2 F_z^2 + 1} \),

where \( \pm \) stands for \( n \) positive or negative and the index \( n \) stands for derivation in normal direction out of the fluid's domain D.

On \( S_w° \), the projection of \( S_w \) on the plane \( Y = 0 \), we have \( \psi_y^{(ii)} = \varepsilon F_x \) for the first order potential.

F. For fixed \( P' \) we have

\[ G = O(1) \text{ with } X \to -\infty; \quad G = O(X^{-1}) \text{ with } X \to +\infty, \]

\[ G_x = O(1) \text{ with } X \to -\infty; \quad G_x = O(X^{-1}) \text{ with } X \to +\infty. \]
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Application of Green's theorem shows that $\psi^{(n)}$ and $\psi$ can be defined subject to the same modes of asymptotic decay, provided the quantity $\delta(X, Y)$ will then turn out to be well behaved.

The symmetry of functions $\psi$ and $\psi^{(n)}$ regarding the lateral coordinate $Y$ will be taken for granted by the symmetry of ship sections and tank profile.

We should bear in mind that [3] for the function $\psi$ existence as a harmonic function is, if at all, guaranteed only in domain $D_0$, but not necessarily in $D^\circ$. However, low order approximations have been derived [3] which are found to exist in the whole interior of $D^\circ$. As for the moment we seek terms up to second degree only, we shall in the following formulate the problem for the domain $D^\circ$ with boundaries known a priori in favor of a less intricate analysis, and derive approximate solutions to this auxiliary problem by perturbation techniques.

Then, for point $P$ within $D^\circ$, we may apply Green's theorem to functions $\psi$ and $G$ to find a representation of $\psi(P)$ as

$$\psi(P) = \frac{1}{4\pi} \int_S \left( \psi_n(P') G(P, P') - \psi(P') G_n(P, P') \right) ds'$$  \hspace{1cm} (1.1)

where the closed boundary $S'$ is composed of $S_w^\circ$, $S_f^\circ$, $S_r$, $S_i$, $S_b^\circ$, $S_e$, and the subscript $n$ stands for normal derivative outward in $P'$ space.

From conditions $C.$ and $D.$ we may conclude that the contributions of $S_i$, $S_r$ and $S_b^\circ$ can be omitted on the right hand side.

The integral over $S_e^\circ$, where the normal derivative is in $Z$ direction, may be transformed by integration regarding $\xi$ and use of $B.8$:

$$\frac{1}{4\pi} \int_{S_e^\circ} (\psi_\xi G - \psi G_\xi) ds = \frac{1}{4\pi} \int_{S_e^\circ} \left( \psi_\xi G - \psi G_\xi \right) \xi \times x_e d\eta + \frac{1}{4\pi} \int_{S_e^\circ} \delta(\xi, \xi) G ds - \frac{1}{4\pi} \int_{L_p} \left( \psi_\xi G - \psi G_\xi \right) ds$$  \hspace{1cm} (1.2)

where the line integral around the ship's load waterline $L_p$ has to be understood in counterclockwise sense when viewed from above (compare (19) [3]).

The line integral has been thoroughly investigated by Yim [5]. We shall find, however, that it is pertinent to merge it with a similar term from the wetted surface $S_w^\circ$. If now we assume the functions $\psi$ and $\psi_\xi$ uniformly bounded for $X \leq X_e$, then, due to the finite size of $S_e$ with conditions $F.$, we may infer that the contribution of $S_e$ becomes insignificant as we let $X_e$ tend to $-\infty$. Similarly: If $\psi$ and $\psi_\xi$ tend to zero with $X \to -\infty$, then, due
to boundedness of $G$ and $G_x$, the contribution of $S_n$ may be neglected with $\lambda_n$ becoming large. But the contributions of $S_n$ and $S_\theta$ must be independent of position $X_n, X_\theta$ in as much as the contribution of the defect $\delta(X,Y)$ may be neglected. Considering higher order terms, however, we will see that independence from $X_\theta$ cannot be assumed in general.

For the first integral in (1.2) over the wetted surface $S_w^\circ$, we shall make the assumption that condition $E_\epsilon$ for $\psi$ holds even for parts of the hull not included in $S_w$ up to the undisturbed free surface, so that we may substitute:

$$\psi_n dS = \varepsilon F_X / \sqrt{ \varepsilon^2 F_X^2 + \varepsilon^2 F_\theta^2 + 1 } dS_w^\circ$$  (1.3)

observing that

$$dS_w^\circ = \sqrt{ \varepsilon^2 F_X^2 + \varepsilon^2 F_\theta^2 + 1 } dS_w^\circ$$  (1.4)

For the second integral over $S_w^\circ$ we substitute the actual components of the normal vector as

$$\left\{ -F_X, r', F_\theta \right\} / \sqrt{1 + \varepsilon^2 F_X^2 + \varepsilon^2 F_\theta^2}$$  (compare [3](18))

and thereby have

$$\frac{-1}{4\pi} \int_{S_w^\circ} \psi G_n dS' = \frac{-1}{4\pi} \int_{S_w^\circ} \psi (\varepsilon \psi') \varepsilon F(\xi, \zeta) \left( \varepsilon F_X (G_x^+ + G_x^-) + \varepsilon F_\theta (G_\theta^+ + G_\theta^-) + \varepsilon F_\theta (G_\theta^+ - G_\theta^-) - G_\theta^+ + G_\theta^- \right) d\xi d\zeta$$  (1.5)

where $I$ stands for $\eta$ positive or negative.

By partial integration regarding $\xi$ and $\zeta$, observing Laplace equation for $G$ as stated in A, and making use of the fact that $F = 0$ at the integration limits if $\zeta < 0$, we then have:

$$\frac{-1}{4\pi} \int_{S_w^\circ} \psi G_n dS' = \frac{1}{4\pi} \int_{S_w^\circ} F(\xi, \zeta) \left( \psi (G_x^+ + G_x^-) + \psi (G_x^+ - G_x^-) d\xi d\zeta \right.$$  (1.6)

$$- \frac{1}{4\pi} \int_{S_w^\circ} F(\xi, \zeta) \psi (G_\theta^+ + G_\theta^-) d\xi d\zeta + \frac{1}{4\pi} \int_{S_w^\circ} \frac{1}{4\pi} \int_{S_w^\circ} \psi (G_\eta^+ - G_\eta^-) d\xi d\zeta - \frac{1}{4\pi} \int_{S_w^\circ} d\xi d\zeta$$  (1.5)

We can now transform the line integral, obtained previously, as
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\[
-\frac{1}{4\pi x_0} \int (\psi G - G \psi) d_\xi = -\frac{1}{4\pi y_0} \int (\psi (G^+ + G^-) - \psi (G_x^+ + G_x^-)) F_x (\xi, 0) d \xi
\]

\[
= -\frac{1}{4\pi y_0} \int \left( \psi_x (G^+ + G^-) - \psi (G_x^+ + G_x^-) \right) F_x (\xi, 0) d \xi
\]

(1.7)

and then combine all components, observing B. for G, as:

\[
\psi = -\frac{e}{4\pi} \int \int F(\xi, \zeta)(G^+ + G^-) d_\xi d_\zeta + \frac{1}{4\pi y_0} \int \int G(\xi, \eta) G d_\xi d_\eta +
\]

\[
+ \frac{e}{4\pi y_0} \int F(\xi, 0) \psi_x (G^+ + G^-) d_\xi - \frac{e}{4\pi} \int \int \left( \psi_x (G^+ + G^-) + \psi (G_x^+ + G_x^-) \right) d_\xi d_\eta
\]

\[
+ \left| \int_0 ^1 \int \left( \psi (\xi, \eta, 0) G - \psi (\xi, \eta, 0) G + \int \int_\eta ^0 \left( \psi_x (\xi, \eta, \zeta) G - \psi (\xi, \eta, \zeta) G \right) d_\xi d_\eta \right| d_\xi d_\eta
\]

\[
+ \frac{1}{4\pi} \int \int \psi (\xi, \epsilon F(\xi, \zeta), \zeta) \left( (G^+ - G^- + \epsilon \frac{\partial G}{\partial \epsilon} \right) d_\xi d_\zeta
\]

(1.8)

If we here neglect the contributions of \( S_0 \) and \( S_e \), the remaining expression, proper behavior of \( \delta (X, Y) \) assumed, really makes this omission legitimate due to the properties stated under F. for the function G.

So far, we have not used any considerations regarding smallness of \( \epsilon = B/L \). We should note that G is defined even for \( \eta = 0 \), i.e., for \( P' \) within the ship's hull, and is well behaved there, if \( P \) is not too close to \( P' \). By development in Taylor series regarding \( \epsilon F \) we therefore may infer that

\[
G_x^+ = -\epsilon F \cdot G_{\eta \eta}^- + O (\epsilon^2)
\]

and

\[
G_x^- = +\epsilon F \cdot G_{\eta \eta}^- + O (\epsilon^2)
\]

(1.9)

which shows that the factor of \( \psi \) in the last integral is small at least of order \( \epsilon^2 \).

In general we have

\[
G_x^+ + G_x^- = G_x (\xi, 0, \zeta) + O (\epsilon^2)
\]

\[
G_x^+ + G_x^- = G_x (\xi, 0, \zeta) + O (\epsilon^2)
\]

(1.10)
If we now assume an expansion
\[ \varphi = \varepsilon \varphi^{(1)} + \varepsilon^2 \varphi^{(2)} + O(\varepsilon^3) \]
\[ \delta = \varepsilon \delta^{(1)} + \varepsilon^2 \delta^{(2)} + O(\varepsilon^3) \] (1.11)
inserting into Green's formula and collecting terms of equal order in \( \varepsilon \), we find that \( \varphi^{(n)} \) is just the expression from Michell's theory with \( \delta^{(n)} = 0 \) and no contributions from \( S_a \) and \( S_e \).

For examination of \( \varphi^{(2)} \) we must go into the nature of \( \delta^{(2)}(X,Y) \). From [2], page 464, (10.12) we find with \( p = \text{const.} \) and replacing \( \delta/\partial t \) by -\( \delta/\partial X \) and \( g \) by \( \gamma \) for our nondimensional representation:
\[ \gamma \psi^{(2)} + \psi^{(2)} = \delta^{(2)}(X,Y) = (\text{grad} \, \varphi^{(n)})^2 + \frac{1}{\gamma} \psi^{(n)}(\gamma \psi^{(n)} + \psi^{(n)})_X \] (1.12)

(It should be observed that only the local component of \( \varphi^{(n)} \) contributes to the expression in brackets in the second term due to structure of \( G \), see Appendix). Sisov's expression corresponding to (1.12) is incorrect.

From the decay of \( G \) and its derivatives as \( O(X^{-1}) \) for \( X \to +\infty \), causing the same mode of decay for \( \varphi^{(n)} \), it may be seen that \( \delta^{(2)} = O(X^{-2}) \) for \( X \to +\infty \) and this means that \( \psi^{(n)} = O(X^{-1}) \) ahead of the ship and the contribution of \( S_a \) to the Green's formula expression may be neglected. If now we can assume that the potential
\[ \psi = \frac{1}{4\pi} \int \delta \cdot G \cdot dS' \] and its X-derivative are
uniformly bounded for \( X \leq X_e \), and to prove this for not too peculiar \( \psi^{(n)} \) should be possible with moderate effort, then we may drop the contribution of \( S_e \) as well as \( X_e \) tends to infinity and may finally write:
\[ \psi^{(2)} = \psi^{(2)} + \psi \]
\[ \psi^{(2)}(X,Y,Z) = \frac{2}{4\pi} \int F(\xi,\zeta) \left[ \psi^{(n)}_X(\xi,\epsilon F,\xi) G_{\xi}(\xi,0,\zeta) + \psi^{(n)}_Z(\xi,\epsilon F,\xi) G_{\xi}(\xi,0,\zeta) d\xi d\zeta + \right. \]
\[ + \frac{2}{4\pi} \int F(\xi,0) \psi^{(n)}_{XX} G(\xi,0,0) d\xi \] (1.13)
The second order potential is thus produced by
1. A distribution of doublets with moment corresponding
to deviation of local first order flow relative to
the ship from uniform parallel flow (giving rise to
the Michell-distribution) times local volume of the
ship.
2. A distribution of sources over the plane \( Z = 0 \) the
density of which is essentially the time-derivative
in an inertial system of dynamic pressure (save a
contribution of the local flow components in the
vicinity of the ship).
3. A line distribution of sources around the ship's
load waterline of output corresponding to local
breadth times wave slope in \( X \)-direction along the
ship's contour according to linear theory.

One should observe that the potential appears only in
derivative form. On the other hand, no differentiability
of the hull surface function \( F \) is required to
make the expression for \( \psi^{(2)} \) meaningful.

For numerical evaluation, the following approxima-
tions are made:
(1) calculating the potential from a distribution \( \delta(X,Y) \)
extending over the entire undisturbed free surface
\( Z = 0 \) including the waterplane area,
(2) inserting the flow components calculated for the
plane \( \eta = 0 \) rather than on the hull surface.

Both these steps require continuation of the flow
potential and some of its derivatives into the domain
occupied by the ship. This is achieved by extension of
the corresponding Fourier series in \( \eta \). In so far as
these series would not converge for \( \eta = 0 \) in the
usual sense, - i.e., for example the series for \( \psi_{\gamma} \),- we
treat them as generalized functions. The second term
of \( \delta(X,Y) \) in (1.12) will in general become singular at
bow and stern; however, for a symmetric hull we will
find that it can be left out for calculation of wave
resistance.

The error involved with above modifications will
in general be of higher order in \( \varepsilon \) than the terms to be
determined, nevertheless, it should be checked for com-
ponents not uniformly bounded in the extended domain.

It should be noted that the expression (13) for \( \psi^{(2)} \)
can be retransformed by partial integration in \( \xi \) and
\( \xi \) to a representation by a source distribution over \( S_w \),
- eliminating the line integral, - so that
We should observe that the source density corresponds to the change of internal flow with coordinate rather than to the normal velocity.

Formula (1.15) may be compared with [3] (43), where the influence of trim and sinkage is included. We see that all line integrals presented there can be eliminated under validity of our assumptions.

2. Determination of wave resistance.

Having thus selected a model for the approximation of the flow, there is a decision to be made for definition of wave resistance to the corresponding degree of approximation. Three different approaches may be considered:

(a) Integrate pressure components over the wetted part of the hull bounded by the calculated wave profile, retain only terms up to third order.

(b) Start with expressions for the energy flow through a vertical plane behind the ship, as given in [2] (8.6) page 460, evaluate these for approximate flow, using wave contour from this approximate flow.

(c) Consider the approximate second order flow to be physically real in the domain $D^e$. Consider a closed surface, part of which is the wetted hull; from the fact that momentum in the enclosed volume $D$ should not change with time, we infer that action of pressure on the hull can be expressed through flow of generalized momentum across the rest of the surface.

It can be shown easily that for the linearized flow model one and the same expression for the resistance can be derived by either approach - (see however the objections raised by Sharma [9]). Up to third order, however, (a) and (c) should only give equivalent expressions $R_a$ and $R_c$, if the boundary condition on the hull is already met exactly by the approximate flow, as otherwise we may have substantial flux of momentum into the ship's interior. The formula for (b) was derived under assumption of a free surface under constant pressure and composed of streamlines. For a second order flow, it is unrealistic to maintain this assumption. We should therefore expect that resistance $R_b$, calculated by this formula applied to the approximate flow, could, even in a nonmonotonic way, depend on the location $X_e$ of the vertical control plane where data are taken.
But $R_c$, derived by approach (c), should be independent of choice of domain $D$. We shall select $D^o$, the domain bounded above by the undisturbed free surface as described before. Due to conservation of momentum we have for surface integrals enclosing any domain $D$ of the flow (compare [2], [7]):

$$\rho \int \left\{ \frac{\vec{v} \cdot \vec{n}}{2} - \frac{(\vec{v} \cdot \vec{n}) \cdot \vec{v}}{2} \right\} \, ds = 0 \quad (2.1)$$

where $\vec{v}$ may be the flow vector in any system of reference either at rest or in uniform translatory motion, $\vec{n}$ be the unit normal vector directed outwards.

If we now select

$$\vec{v} = \{ \varphi_x, \varphi_y, \varphi_z \} = \{ \psi_x, \psi_y, \psi_z \} \cdot \mathbf{c}$$

and define $R_c$ as $X$-component of

$$\vec{R} = \rho \int \left\{ \frac{\vec{v} \cdot \vec{n}}{2} - \frac{(\vec{v} \cdot \vec{n}) \cdot \vec{v}}{2} \right\} \, dS,$$  \quad (2.2)

where integration has to be performed over $\zeta_{w'}$, the hull surface up to $Z = 0$, then we have from (1.15), returning to nondimensional quantities, with $\vec{c}_x$ as unit vector in $X$-direction,

$$R_c = \frac{\vec{R}_c \cdot \vec{c}_x}{\rho \vec{c}_y (L/2)} = \gamma_o \int \int \psi_x \psi_z \, d\xi \, d\eta \gamma_o \int \int \psi_x \psi_y \, d\xi \, d\eta \frac{\psi_z^2 - \psi_x^2}{2} \, d\eta \, d\xi$$

$$+ \gamma_o \int \int \psi_x \psi_z \, d\xi \, d\eta + \gamma_o \int \int \psi_y \psi_z \, d\xi \, d\eta \frac{\psi_y^2 + \psi_z^2 - \psi_x^2}{2} \, d\eta \, d\xi \quad (2.3)$$

Reference to conditions $C$, and $D$, shows, that the surface $S_{t}$, $S_{l}$ and $S_{b}$ may be left out. The integral over $S_{f}$ may be transformed to line integrals along the boundaries and an integral containing the function $\delta(X,Y)$ in a similar way as was done for the potential (1.2). The contribution from $S_{a}$, including the line integral from $S_{f'}$, tends to zero with $X \to \infty$ due to $F$. Thus, we are left with

$$R_c = \int \Delta (\zeta, \eta) \psi_x (\zeta, \eta, 0) \, d\xi \, d\eta - \oint \frac{\psi_x^2 (\zeta, \pm f(\zeta, 0), 0) \, dL}{2} + \int \int \left\{ \frac{\psi_x^2 (x, \eta, 0)}{2} + \int \psi_y (x, \eta, \zeta) + \psi_z (x, \eta, \zeta) - \psi_x (x, \eta, \xi) \, d\zeta \, d\eta \right\} \, d\eta \quad (2.4)$$
where the line integral over the load waterline \( L_p \) is again in counterclockwise direction when viewed from above.

Now, there is no reason to assume that the contribution from \( S_i^p \), though bounded in magnitude, should tend to a definite limit with \( X_c \to -\infty \), nor can we postulate this for the contribution of the region \( X = X_c \). It is only by some property of the Green's function \( G \) involved that we shall be able to evaluate the contribution of \( \delta(X,Y) \) to the resistance by an integration on \( S_i^* \) in the vicinity of the ship only. - We shall now look for a relation between the quantities defined as \( R_c \) and \( R_a \). To achieve this, we add an expression to the integrand of (2.2) which has no component in direction of \( \hat{e}_x \). We set

\[
R_c = R_c + \hat{e}_x \rho \int \left( ((\vec{v} \cdot \hat{c}) \cdot \vec{n}) \right) dS = R_c + \rho \int \left( \left( (\vec{v} \cdot \hat{c}) \cdot \vec{n} \right) - (\vec{c} \cdot \vec{n}) \vec{v} \right) dS
\]

and therefore, with \( \vec{c} \cdot \vec{n} = (\vec{v} \cdot \vec{n}) \) assumed even for the first order flow,

\[
R_c = \int \rho \left( \hat{e}_x \cdot \vec{n} \right) \cdot \left( (\vec{c} \cdot \vec{n}) - (\vec{v} \cdot \vec{n}) \right) / 2 \right) dS. \tag{2.6}
\]

But this is just \( R_a \), the resistance defined from pressure integration over the hull, for the nonstatic pressure is \( \rho ((\vec{v} \cdot \vec{c}) - (\vec{v} \cdot \vec{v}) / 2) \) and we thus have

\[
R_a = R_c + \rho \int \varphi_x ((\vec{c} \cdot \vec{n}) - (\vec{v} \cdot \vec{n})) dS \tag{2.7}
\]

To \( R_c \) as defined above, we now add an appropriate correction for the influence of the wave profile along the waterline, as only the wetted part of the hull can experience pressure from the fluid, and then define the quantity obtained as "third order wave resistance". Now, up to second order, pressure is atmospheric pressure plus hydrostatic pressure due to the wave elevation \( \xi \). Integrating the last quantity over dZ dY, the projection of the surface element on the plane vertical to X axis, we find a correction as

\[
\Delta R = \rho g \int \varphi \left( (Z - \zeta) \right) dZ d\eta = \rho g \int \frac{\xi^2(\xi, \eta)}{2} d\eta. \tag{2.8}
\]

As now the perturbation procedure gives the first order wave elevation [2] as

\[
\xi(\xi) = \frac{c}{q} \varphi_x \quad i.e. \quad \frac{\xi(\xi)}{L/2} = \frac{1}{\gamma_0} \psi_x(X, Y, 0), \tag{2.9}
\]
adding this in nondimensional form to expression (2.4), we see that this correction just cancels the line integral around \( L_p \), which we, therefore, can happily discard \([8]\).

A further simplification will be made by extending the integration of \( \delta \) over the whole plane \( \xi = 0 \), \(-T < \eta < +T\), \( \xi < X_e \) which means an error of order \( \varepsilon^2 \), as the waterplane area is of order \( \varepsilon \). Inserting now

\[
\psi = \varepsilon \psi \; ^{(1)} + \varepsilon^2 \psi \; ^{(2)} , \quad \delta = \varepsilon^2 \delta \; ^{(2)} , \quad R = \varepsilon^2 R \; ^{(2)} + \varepsilon^3 R \; ^{(3)} \]

we have

\[
R \; ^{(2)} = \int \left\{ \frac{1}{2} \psi \; ^{(1)} X_e, \eta, 0 + \int_0^\infty \frac{\psi \; ^{(0)} Y + \psi \; ^{(0)} Z - \psi \; ^{(0)} X}{\xi} d \xi \right\} d \eta \quad (2.10)
\]

\[
R \; ^{(3)} = R \; ^{(3)} + R \; ^{(j)} \quad (corresponding \ to \ the \ partition \ \psi \; ^{(2)} = \psi \; ^{(1)} + \psi \; ^{(2)})
\]

with

\[
R \; ^{(1)} = \int \left\{ \frac{1}{2} \psi \; ^{(1)} X_e, \eta, \xi \; ^{(1)} X_e + \int_0^\infty \psi \; ^{(0)} Y \; ^{(1)} Y + \psi \; ^{(0)} Z \; ^{(1)} Z - \psi \; ^{(0)} X \; ^{(1)} X d \xi \right\} d \eta \quad (2.11)
\]

\[
R \; ^{(3)} = -\int \int \int \delta \; ^{(2)} (\xi, \eta, \xi) \; ^{(1)} X_e d \xi d \eta + \int \left\{ \frac{1}{2} \psi \; ^{(1)} X_e, \eta, \psi \; ^{(0)} Y + \psi \; ^{(0)} Z - \psi \; ^{(0)} X \right\} d \eta \quad (2.12)
\]

3. Resistance due to additional singularities within the hull.

Let us now, only to save labor in writing down formulas, assume that the depth of the tank is large enough that we may put \( H = \infty \). If \( \psi \; ^{(1)} \) can at \( X = X_e \) be represented by a system of free waves - we omit terms nonsymmetric in \( Y \) for reasons of simplicity - as

\[
\psi \; ^{(1)} \; ^{\text{free}} = \sum_{\nu = \infty}^\infty \left( A \; ^{(1)} \nu \cos (W \; ^{(1)} \nu X) + B \; ^{(1)} \nu \sin (W \; ^{(1)} \nu X) \right) \cdot e^{K \; ^{(1)} \nu Z} \cos (U \; ^{(1)} \nu Y) \quad (2.13)
\]

where \( \Delta U = \pi / (\eta T) \) and \( U \nu = \nu \cdot \Delta U = \sec^2 \theta \nu \sin \theta \nu ; M \nu = 1 + 4 U \nu^2 ; K \nu = (1 + M \nu) / 2 = \sec^2 \theta \nu ; W \nu = \sqrt{K \nu} = \sec \theta \nu ; A \; ^{(1)} \nu = A \; ^{(1)} \nu ; B \; ^{(1)} \nu = B \; ^{(1)} \nu \)

(where \( \theta \nu \) stands for the angle of wave propagation against \( X \) axis), then \([6]\) we can evaluate the integrals in closed form as
Eggers

\[ R^{(2)} = T \sum_{\nu=-\infty}^{\infty} \frac{2 - \cos^2 \theta_\nu}{2} \left( A^{(u)}_\nu + B^{(u)}_\nu \right) / \gamma_\nu \]

(2.14)

(This formula reflects the fact that resistance is essentially equal to average energy in wave components times difference between ship's speed \( c \) and \( X \)-component of group-velocity, divided by \( c \).

If \( \psi^{(2)}_x \) has a corresponding far-field representation:

\[ \psi^{(2)(\text{free})}_x = \sum_{\nu=-\infty}^{\infty} \left\{ A^{(2)}_\nu \cos(W_\nu \gamma_\nu X) + B^{(2)}_\nu \sin(W_\nu \gamma_\nu X) \right\} e^{K_x \nu \gamma_\nu^2} \cos(U_\nu \gamma_\nu Y) \]

(2.15)

then \( R^{(3)}_x \) as interference between both systems can be written down directly as

\[ R^{(3)}_x = T \sum_{\nu=-\infty}^{\infty} (2 - \cos^2 \theta_\nu) \left( A^{(u)}_\nu \cdot A^{(u)}_\nu + B^{(u)}_\nu \cdot B^{(u)}_\nu \right) / \gamma_\nu \]

(2.16)

For evaluation of (2.14) and (2.16) we have to keep in mind that for \( \eta = 0 \) the Green's function \( G \) has a representation for \( \xi \ll X \) as system of free waves like

\[ G_x \sim \sum_{\nu=-\infty}^{\infty} q_\nu \cos(W_\nu \gamma_\nu (X - \xi)) e^{K_x \nu \gamma_\nu^2} \cos(U_\nu \gamma_\nu Y) \]

(2.17)

with \( q_\nu = q_{-\nu} = -8 \pi \cdot K_x \cdot \gamma_\nu / (M_\nu \cdot T) \)

(see Appendix) and that we have:

\[ \psi^{(1)} = -\frac{2}{4\pi} \iint_{S_{xx}^{**}} F(\xi, \zeta) \cdot G_x \, d\xi \, d\zeta \]

(2.18)

\[ \psi^{(2)}_x = \frac{2}{4\pi} \iint_{S_{wx}^{**}} F(\xi, \zeta) \left\{ \psi^{(u)}_x G_x + \psi^{(u)}_z G_{\xi x} \right\} d\xi \, d\zeta + \frac{2}{4\pi \gamma_\nu} \int_{F(\xi,0)G_x^{(u)} G_x} \]

\[ = -\frac{1}{2\pi} \iint_{S_{wx}^{**}} \left\{ (F \psi_x)_x + (F \psi_z)_z \right\} G_x \, d\xi \, d\zeta \]

(2.19)

and therefore

\[ A^{(u)}_\nu = \frac{g_\nu}{2\pi} \iint_{S_{wx}^{**}} F(\xi, \zeta) \sin(W_\nu \gamma_\nu \xi) \cdot e^{K_x \nu \gamma_\nu^2} \cdot W_\nu \gamma_\nu \, d\xi \, d\zeta \]

(2.20)
Eggers

\[ B_{v}^{(1)} = -\frac{g_{v}}{2\pi} \int_{S_{w}^{**}} F(\xi, \zeta) \cdot \cos(W_{v}, \gamma_{0}, \xi) \cdot e^{K_{v}r_{0}^{2}} \cdot W_{v} \gamma_{0} d\xi d\zeta \]  
(2.21)

\[ A_{v}^{(2)} = \frac{g_{v}}{2\pi f_{r}} \int_{S_{w}^{**}} F(\xi, 0) \cdot \psi_{x}^{(2)} \cdot \sin(W_{v}, \gamma_{0}, \xi) d\xi + \frac{g_{v}}{2\pi f_{r}} \int_{S_{w}^{**}} F(\xi, 0) \cdot \psi_{xx}^{(2)} \cdot \sin(W_{v}, \gamma_{0}, \xi) d\xi \]  
(2.22)

\[ B_{v}^{(2)} = -\frac{g_{v}}{2\pi f_{r}} \int_{S_{w}^{**}} F(\xi, 0) \cdot \psi_{xx}^{(2)} \cdot \cos(W_{v}, \gamma_{0}, \xi) d\xi . \]  
(2.23)

The above expressions can in general be evaluated in closed form for mathematical elementary hulls, save the contributions of local flow \( \psi^{(1)} \) to the integrands, where however the \( V \)-integration may be interchanged with closed-form \( \xi, \zeta \) integration.

4. Resistance due to additional singularities at undisturbed free surface.

Consider a strip of width \( d\xi \) extending from \( \eta = -T \) to \( \eta = T \) at \( \zeta = 0 \) with ordinate \( \chi = \xi \). Assume that a Fourier expansion for \( \delta(\xi, \eta) \) holds as

\[ \delta^{(2)}(\xi, \eta) = \sum_{n=-\infty}^{\infty} \delta_{v}(\xi) \cdot \cos(U_{v} \gamma_{0}, \eta) \]  
(2.24)

If \( X_{v} \ll \xi \), this strip will contribute to \( \psi_{xx}^{(2)} \) by

\[ d\psi_{xx}^{(2)} = \int_{-T}^{T} \frac{\delta^{(\text{free})}}{4\pi f_{r}} G_{x}^{(\text{free})}(\xi, \eta, 0, X, Y, Z) d\eta d\xi \]  
(2.25)

with

\[ G_{x}^{(\text{free})} = \sum_{n=-\infty}^{\infty} g_{v} \cdot e^{K_{v}r_{0}^{2}} \cdot \cos(W_{v}, \gamma_{0}(X-\xi)) \cdot \cos(U_{v} \gamma_{0}, Y) \cdot \cos(U_{v} \gamma_{0}, \eta) \]  
(2.26)

\[ ( + \text{terms odd in } \eta \text{ not needed here}) \]

where

\[ g_{v} = g_{v} - 4\pi K_{v} r_{0}^{2} / (M_{v} \cdot T) \]  
(2.27)

(compare (A.1)). We then have:
\[ d \psi^{(1)}_{\xi, \tau} = 2T \sum_{\nu = -\infty}^{\infty} \delta_{\nu} \cdot K_{\nu} \left( \frac{1}{M_{\nu} \cdot T} \right) \cdot e^{K_{\nu} \xi \tau} \]
\[
\times \left\{ \cos(W_{\nu \xi} \xi) \cos(W_{\nu \xi} \tau) + \sin(W_{\nu \xi} \xi) \sin(W_{\nu \xi} \tau) \right\} \cdot \cos(U_{\nu \xi} \eta) \, d\xi 
\]

(2.28)

If now for \( \psi^{(1)}_{\xi} \) as well only free waves are significant at \( \tau = \xi \), i.e., if we have:

\[
\psi^{(1)}_{\xi} = \psi^{(1)\text{(free)}}_{\xi} = \sum_{\nu = -\infty}^{\infty} \left\{ A^{(1)}_{\nu} \cos(W_{\nu \xi} \xi) + B^{(1)}_{\nu} \sin(W_{\nu \xi} \xi) \right\} e^{K_{\nu} \xi \tau} \cos(U_{\nu \xi} \eta) 
\]

(2.29)

then \( d \psi^{(1)}_{\xi, \tau} \) will make up a contribution to \( R^{(1)}_{\xi, \tau} \) as

\[
dR^{(1)}_{\xi, \tau} = T \sum_{\nu = -\infty}^{\infty} \delta_{\nu} \left( 2 - \cos^2 \Theta_{\nu} \right) K_{\nu} M_{\nu} \left( A^{(1)}_{\nu} \cos(W_{\nu \xi} \xi) + B^{(1)}_{\nu} \sin(W_{\nu \xi} \xi) \right) \, d\xi 
\]

(2.30)

But from (2.13) we may derive that

\[
K_{\nu} / M_{\nu} = K_{\nu} / (2K_{\nu} - 1) = 1 / (2 - \cos^2 \Theta_{\nu}) , 
\]

(2.31)

thus

\[
dR^{(1)}_{\xi, \tau} = 2T \sum_{\nu = -\infty}^{\infty} \delta_{\nu} \left\{ A^{(1)}_{\nu} \cos(W_{\nu \xi} \xi) + B^{(1)}_{\nu} \sin(W_{\nu \xi} \xi) \right\} 
\]

(2.32)

\[
= \int_{\eta = -T}^{T} \delta(\xi, \eta) \psi^{(1)\text{(free)}}_{\xi} (\xi, \eta, 0) \, d\eta \, d\xi 
\]

This is not yet the whole contribution of the strip to \( R^{(1)}_{\xi, \tau} \), however; from (2.12) we have to add:

\[
dR^{(1)}_{\xi, \tau} = -\int_{\eta = -T}^{T} \delta(\xi, \eta) \psi^{(1)}_{\xi} (\xi, \eta, 0) \, d\eta \, d\xi 
\]

(2.33)

This leads to the simple result

\[
R^{(1)}_{\xi, \tau} = \int_{-\infty}^{\infty} \int_{-T}^{T} \delta^{(1)}(\xi, \eta, 0) \tilde{\psi}_{\xi, \tau} (\xi, \eta, 0) \, d\eta \, d\xi 
\]

(2.34)

with \( \tilde{\psi}_{\xi, \tau} = \psi^{(1)}_{\xi} - \psi^{(1)\text{(free)}}_{\xi} \)

(2.35)

The potential \( \tilde{\psi} \) would correspond to the solution of the first order boundary value problem if we had postulated waves traveling ahead of the ship instead of aft the
ship. For a ship symmetrical to the midship section we may insert $\psi(X) = \psi(-X)$.

The above integral (2.34) will have significant contributions to the vicinity of the ship, as $\delta$ has strong decay ahead and the factor $\psi_X$ shows a decay aft. The overwhelming contribution should therefore come from the rhombe-shaped region bounded by a Kelvin angle drawn from the bow and an opposite angle from the stern.

The expression for $R_z^{(3)}$ could have been derived directly as the Lagally force of the wave field due to the surface disturbance $\delta(\xi, \eta)$ acting on the singularities creating the first order flow field of the ship. It would, therefore, have been found by Sisov under use of proper radiation condition. For the case of a nonsubmerged body, we felt that formal application of Lagally's law even for higher order contributions deserved caution. Inserting (1.13) we have

$$R_z^{(3)} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (\text{grad} \psi''')^2 + \psi''(y \psi'' + \psi''')_y / Y \right] \left[ \psi'' \psi'' + \psi'''' \right] dX dY$$

(2.36)

In peculiar for a symmetrical hull, where $\psi'' + \psi''''$ is odd and $\psi''''$ is even, we have:

$$R_z^{(3)} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{grad} \psi''')^2 \psi''(-X) dX dY = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{grad} \psi''')^2 \psi''''(-X) dX dY$$

$$=-\frac{1}{\kappa_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{grad} \psi''')^2 \psi''''(-X) dX dY$$

(2.37)

If now for a symmetrical hull we have:

$$\psi''''(X,Y,0) = \sum_{\nu=-\infty}^{\infty} \alpha_\nu(X) \cos(U \cdot \gamma \cdot Y) \quad \text{with } \alpha_\nu = \alpha_{-\nu}$$

$$\psi'''(X,Y,0) = \sum_{\nu=-\infty}^{\infty} \beta_\nu(X) \sin(U \cdot \gamma \cdot Y) \quad \text{with } \beta_\nu = \beta_{-\nu}$$

$$\psi''(X,Y,0) = \sum_{\nu=-\infty}^{\infty} \gamma_\nu(X) \cos(U \cdot \gamma \cdot Y) \quad \text{with } \gamma_\nu = \gamma_{-\nu}$$

(2.38)

where according to (2.18) and (A.1) the coefficients $\alpha_\nu$, $\beta_\nu$, and $\gamma_\nu$ depend on hull geometry given by $Y = \rho \cdot \Gamma(X,Z)$ through the relations...
\[ \alpha_v = \frac{\partial H_v}{\partial X}, \quad \beta_v = U_v y_v H_v, \quad \gamma_v = \frac{\partial H_v}{\partial Z} \quad (2.39) \]

with the function \( H_v(X,Y,Z) \) given by

\[
H_v(X,Y,Z) = \frac{2 K_v r_0}{T \cdot M_v} \int \left[ \frac{\text{sign}(X - \xi)}{\xi} \right] e^{K_v \xi} F(\xi, \bar{\xi}) \cos(W_v y_v(X - \xi)) d\xi d\bar{\xi}
\]

\[
= -\frac{2 r_0}{\pi T} \int \int \frac{\text{sign}(X - \xi)}{\xi} F(\xi, \bar{\xi}) e^{-U_v |X - \xi|} \left[ \cos(W_v y_v(\xi - \bar{\xi}) - U \sin(W_v y_v) \right] \frac{U dV}{U^2 + V^2} \quad (2.40)
\]

and \( U_v, M_v, W_v, K_v \) and \( U \) as given by \( (A.4) \), then we can express

\[ R_2^{(3)} = R_2^{(30)} \cdot 4 L / (\rho \beta^3 c^2) \]

as

\[
R_2^{(3)} = -\frac{T}{\delta_0} \int \sum_{\mu = -\infty}^{\infty} \sum_{\lambda = -\infty}^{\infty} \left[ \alpha_\mu(X) \alpha_\lambda(X) - \beta_\mu(X) \beta_\lambda(X) + \gamma_\mu(X) \gamma_\lambda(X) \right] y(-X) +
\]

\[+ \left[ \alpha_\mu(X) \alpha_\lambda(X) + \beta_\mu(X) \beta_\lambda(X) + \gamma_\mu(X) \gamma_\lambda(X) \right] y(-X) ] dX \quad (2.41)
\]

due to the Fourier orthogonality relations for \( Y \) integration, where the \( X \) integral may be truncated soon after \( X \) exceeds 1 in absolute value, convergence of \( (2,41) \) assumed.

The above formula can easily be extended to the case of infinite tank width with \( T \to \infty \); however, for practical evaluation it is recommended to consider \( T \) as inverse of spacing in integration by trapezoidal rule and let \( T \) be just large enough, dependent on \( X \), that for \( \xi = \pm X \) the actual wave pattern is well within \( |Y| < T \), i.e., that no tank effect can be felt.

For actual calculations we have to reintroduce dimensions; we have

\[ R_2^{(3)} = (B/L)^3 \cdot \rho \cdot c^2 \cdot (L/2)^2 \cdot R_2^{(3)} = \rho c^2 B^3 / (4L) \cdot R_2^{(3)} \]

\[ R_2^{(2)} = (B/L)^2 \cdot \rho \cdot c^2 \cdot (L/2)^2 \cdot R_2^{(2)} = \rho c^2 B^2 / 4 \cdot R_2^{(2)} \quad (2.42) \]

(compare \( (2.3) \), \( (2.10) \), \( (2.11) \) \( (2.12) \)). -- where \( R_2^{(3)} \) and \( R_2^{(2)} \) are the actual third and second order resistance components.
Fig. 2


Ship with parabolic waterlines
Draft/Length = 1/20

\( \gamma_0 = 0.5 \, \rho_n^{-2} = 10 \)

Percentage of resistance from cut at
Y = \( \gamma \)
truncated at X = \( x \).

Draft/Length = 1/20
Summary.

With the above analytical considerations, an attempt was made to coordinate the intuitive approach of Sisov with the rigorous procedure of Wehausen. Some simplifications allowed, we found that even the latter leads to a representation of the second order wave potential by sources only, located on the undisturbed free surface and on the longitudinal centerplane of the ship; in particular all line integrals can be eliminated. Additional resistance can be expressed in terms of first order flow components which determine these singularities. Only a region of the free surface close to the ship need be considered.

Acknowledgement.

Development of the fundamental program for computations was initiated during the author's National Science Foundation Fellowship at the University of Notre Dame and continued under support of Deutsche Forschungsgemeinschaft. The author wishes to express gratitude to Deutsches Elektronen Synchrotron, Hamburg, for generosity in making available their IBM 7044 computer facilities. He also thanks the H-5 Panel of the Society of Naval Architects and Marine Engineers for their continued interest in his work.

Fig. 3

Ratio of Calculated Resistance $R_1(x)$ from Transverse Cut Analysis [6] (Wave Elevation and X-slope) to Asymptotic Value $R_2(x)$

$\kappa_0 = 3.86$

Infinite strut

tank width = 5/6 ship length
Appendix

A Fourier series representation of the source potential in a tank of finite width.

The expression to be presented here has essentially been derived in [6]. We shall confine ourselves to show certain properties which are needed in the foregoing applications.

The expression describes a wave potential of a source of output $+4\pi$, i.e., a singularity like negative inverse distance, in coordinates made dimensionless by ship's half length as introduced under (1). The expression is:

$$ G = 2\pi \sum_{n=1}^{\infty} \left[ g_\nu^{(\text{free})}(X,\xi,\eta) + g_\nu^{(\text{local})}(X,\xi,\eta) \right] \left\{ \cos(U_\nu Y) \cos(U_\nu \eta) \{ 1 + (-1)^n \} + \sin(U_\nu Y) \sin(U_\nu \eta) \{ 1 - (-1)^n \} \right\} \Delta U$$

(A.1)

with $\Delta U = \pi/(y_0 T)$, $U_\nu = \nu \Delta U = \sec^2 \theta \sin \theta$, $g_\nu^{(\text{free})} = -g_\nu^{(\text{free})}$, $W_\nu = \frac{1}{\sqrt{1 + 4 U_\nu^2}}$, $K_\nu = (1 + M_\nu)/2 = \sec^2 \theta$, $W_\nu = \sqrt{K_\nu} = \sec \theta$, and

$$ g_\nu^{(\text{local})} = g_\nu^{(\text{local})} = \frac{1}{\pi} \int_{\nu_0}^{\infty} \left[ e^{-u_\nu^{\prime} |X - \xi|} \cdot \left\{ \left( V \cos(V_\nu Z) - U^2 \sin(V_\nu Z) \right)x \left( V \cos(V_\nu Z) - U^2 \sin(V_\nu Z) \right) \right\} - \delta_\nu(u) \right\} / \left( V(U^2 + V^2) \right) \cdot dV$$

(A.2)

with $U = \sqrt{U^2 + U_\nu^2}$ and $\delta_\nu(0) = 1$ for $\nu = 0$, else $\delta_\nu = 0$.

It is easy to find out by investigation of single terms of the series that the function $G$ is subject to the following conditions:

A. $G_{xx} + G_{yy} + G_{zz} = 0$ provided the corresponding $V$ integrals exist, which is guaranteed for $|X - \xi| > 0$

B. $x G + G_{xx} = 0$ for $Z = 0$
Eggers

\[ G_Y = 0 \text{ for } Y = \pm T \]

\[ G_Z \rightarrow 0 \text{ with } Z \rightarrow -\infty \]

\[ G = 0 (X - \xi)^{-1} \text{ as } X \rightarrow +\infty, \quad G = 0(1) \text{ as } X \rightarrow -\infty, \]

\[ G_X = 0 (X - \xi)^{-1} \text{ as } X \rightarrow +\infty, \quad G_X = 0(1) \text{ as } X \rightarrow -\infty, \]

explicitly shown in [6].

As the structure of \( G \) is symmetric, corresponding relations can be obtained under exchange of \( X, Y, Z \) with \( \xi, \eta, \zeta \).

It remains to be shown that

(i) the expressions for \( G \) and \( G_X \) match in a continuous way at \( X = \xi \).

(ii) for \( |Y - \eta| \leq T, Z \geq 0 \) and \( \xi \leq 0 \), \( G \) and \( G_X \) become singular only for \( X = \xi \), \( Y = \eta \), and \( Z = \zeta \), and that the functions

\[ G^* = G + 1/r \quad \text{and} \quad G_X^* = G_X + (1/r)_X, \]

with \( r = \left\{ (X - \xi)^2 + (Y - \eta)^2 + (Z - \zeta)^2 \right\}^{-1/2} \)

remain finite here: - (convergence of the series for \( \xi + Z \) is shown in [6]). Statement (i) is evident for the function \( G \). Assume for simplicity \( \xi = 0 \).

Then it is sufficient to show that

\[ \frac{K_{\nu}}{M_{\nu}} e^{K_{\nu} Y_e (Z + \zeta)} = \lim_{x \rightarrow 0} \int_{v=0}^{\infty} e^{-K_{\nu} |x|} \left( V \cos(V e x) - U \sin(V e x) \right) \left( V \cos(V e x) - U \sin(V e x) \right) \frac{dV}{U x + V^2} \]

for arbitrary \( U_{\nu} \geq 0 \) and \( Z + \xi \) with

\[ U = \sqrt{V^2 + U^2}, \quad M_{\nu} = \sqrt{1 + 4 U_{\nu}^2}, \quad K_{\nu} = (1 + M_{\nu}) / 2. \]

On the right hand side we may substitute

\[ \frac{1}{2 \pi} \lim_{x \rightarrow 0} \left\{ \int_{v=0}^{\infty} \left( V^2 \cos(V e x) + V e x \right) \frac{dV}{U x + V^2} \right\} \]

+ \[ 2 U^2 V \sin(Y e (Z + \xi)) \]

\[ \int_{v=-\infty}^{\infty} \frac{e^{-U x} V e x}{U x + V^2} \]

\[ = \int_{v=-\infty}^{\infty} e^{i V e x (Z + \xi)} e^{i V e x (Z + \xi)} e^{i V e x (Z + \xi)} \frac{dV}{V - i U^2} \]

\[ + \int_{v=-\infty}^{\infty} \frac{(e^{-U x} X) - e^{-V e x (Z + \xi)} e^{i V e x (Z + \xi)} e^{i V e x (Z + \xi)}}{V - i U^2} \frac{dV}{V - i U^2} \]

(A.5)
The second term in the last expression is zero for any finite X. The last term is o(X). The first term may be written

\[
\lim_{x \to 0} \frac{1}{4\pi} \text{Re} \left\{ \int_{-\infty}^{\infty} \frac{(V-iK\nu)(V+i(M\nu+1)/2)}{(V+iK\nu)(V-i(M\nu+1)/2)} e^{i[V\nu(\zeta+z)-V\nu]\ln |x|} dV \right\} (A.6)
\]

which shows poles of the integrand for \( V = i(M\nu-1)/2 \) and \( V = -i(M\nu+1)/2 = -iK\nu \). By shifting the path of integration downward in the complex plane we can make the integral arbitrary small after splitting off the residuum at \( V = -iK\nu \), thus we finally get

\[
\lim_{x \to 0} \frac{1}{4\pi} \text{Re} \left\{ \frac{(V-iK\nu)(V+i(M\nu+1)/2)}{(V+iK\nu)(V-i(M\nu+1)/2)} e^{i[V\nu(\zeta+z)]} \right\} \frac{1}{4\pi} \frac{K\nu e^{K\nu(V+z)}}{M\nu} (A.7)
\]

q.e.d.

To prove the statement (ii) we start with the representation

\[
\gamma_1 = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-u|x-\xi|} \cos \left\{ (Y-\eta)\cos \theta + (Z-\xi)\sin \theta \right\} d\nu d\gamma (A.8)
\]

\[
\gamma_1 = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-u|x-\xi|} \cos \left\{ (Y-\eta)\cos \theta + (Z-\xi)\sin \theta \right\} d\nu d\gamma (A.9)
\]

where \( r_1 \) corresponds to \( r \) with \( \xi \) under negative sign. - Introducing new variables of integration \( U, V, \tilde{U} \) by

\[
U = u/\gamma_1 ; \quad \tilde{U} = u \cos \theta /\gamma_1 ; \quad V = u \sin \theta /\gamma_1 ; (A.9)
\]

we have

\[
1/r - 1/r_1 = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-U\nu_x |x-\xi|} \cdot 2 \sin (V\nu_x Z) \sin (V\nu_x \zeta) \cos (\tilde{U}\nu_x (Y-\eta)) \frac{d\tilde{U}dU}{d\nu dV} (A.10)
\]

\[
(1/r - 1/r_1) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-U\nu_x |x-\xi|} \cdot 2 \sin (V\nu_x Z) \sin (V\nu_x \zeta) \cos (\tilde{U}\nu_x (Y-\eta)) \frac{d\tilde{U}dV}{d\nu dV \cdot \text{sign}(x-\xi)} (A.11)
\]

Now there is a general law in the theory of Fourier transforms [11] - essentially known as Poisson's summation rule - stating that if the function \( F(y) \) has
a representation
\[ F(y) = \int G(\tilde{U}) e^{i\tilde{U}y} \, d\tilde{U} \quad (A.12) \]
then \( F^*(\delta, y) = \sum_{\nu=-\infty}^{\infty} F(y + \nu \delta) \), provided this series converges, has a representation
\[ F^*(\delta, y) = \sum_{\nu=-\infty}^{\infty} G(U_{\nu}) e^{iU_{\nu}y} \cdot \Delta \tilde{U} \quad (A.13) \]
with \( \Delta \tilde{U} = 1/\delta \), \( U_{\nu} = \nu \cdot \Delta \tilde{U} \)

With \( \delta = T/\pi \), we therefore have the representation
\[
\sum_{\nu=-\infty}^{\infty} \left[ \frac{1}{(X-\xi)^2 + (Y+\nu T-\eta)^2 + (Z+\zeta)^2} - \frac{1}{(X+\xi)^2 + (Y+\nu T-\eta)^2 + (Z+\zeta)^2} \right]^{1/2} = \]
\[
-2\pi \int_{-\infty}^{\infty} e^{-U_{\nu}|X+\xi|} \sin(V_{\nu,\xi}Z) \sin(V_{\nu,\zeta}) / U \cdot dV \cdot \cos(U_{\nu,\xi}(Y-\eta)) \quad (A.14) \]
with \( U_{\nu} = 2\pi \nu/(\xi, T) \); \( U = \sqrt{V^2 + U_{\nu}^2} \).

But the terms under summation are equivalent to corresponding terms in the series for \( G \) \((A.1)\), and it can be seen that after subtraction of these terms the integrals for the coefficients \( q_{\text{local}}^X \) \((A.1)\) converge even in the case \( X = \xi, Z = \zeta \).

The argument for the function \( G_X \) is analogous.
References:


