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Maneuvering Hydrodynamics of Ellipsoidal Forms

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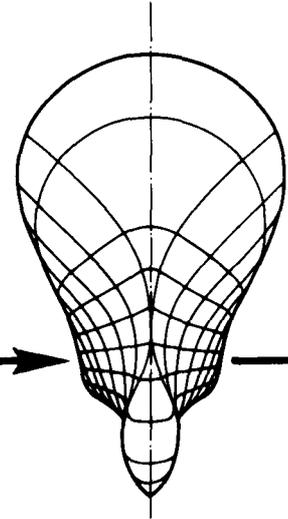
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**Maneuvering Hydrodynamics of
Ellipsoidal Forms**

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MANEUVERING HYDRODYNAMICS OF ELLIPSOIDAL FORMS

by

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Abstract

The hydrodynamical forces and moments acting on a tri-axial ellipsoid moving in an incompressible and inviscid fluid are analysed. The rigid ellipsoid is allowed to move in the most general manner with time dependent velocity and six degrees of freedom. The force and moment expressions are obtained by applying the Lagally theorem to the image singularities system representing the body in the presence of exterior disturbances. First expressions for the Lagally force and moment acting on a maneuvering ellipsoid in an unbounded medium are derived and then these expressions are generalized to include the effect of an exterior source moving in an arbitrary manner. It is also shown how the Lagally expressions for an exterior source can be used to obtain closed form expressions for the hydrodynamical forces and moments acting on a maneuvering ellipsoid in the presence of arbitrary exterior disturbance. The analysis which is based on the application of ellipsoidal harmonics is demonstrated in a simple case of propeller-hull interaction. Here the motion of the ellipsoid is restricted to the major axis and the propeller at the stern is represented by an isolated sink in accordance with Dickmann's model. Practical expressions for the thrust deduction coefficient, wake fraction and propeller induced vibration are then derived for ellipsoidal, spheroidal and spherical hulls.

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1. INTRODUCTION

We are now celebrating the 25th anniversary of the Institut für Schiffbau. This is an occasion to remember Professor Weinblum, the renowned first director of the Institute. Among the many imprints Professor Weinblum left in the field of ship theory were his studies of Maneuvering Hydrodynamics of general ship forms and of ellipsoidal forms in particular. Professor Weinblum advocated the use of ellipsoidal forms as an approximation for general ship forms and his often repeated statement "the ellipsoid is God's gift to naval architects!" (quoted by Newman (1972)) has stimulated many valuable studies. His approach to ship maneuvering extensively used the Lagally theorem of which Professor Weinblum was one of the main promoters and one of the first to realize its power and versatility. For these reasons the present paper which analyses general maneuvering of ellipsoids by employing the Lagally theorem is dedicated to Professor Weinblum's memory in this jubilee year of the Institut für Schiffbau.

Due to the ever increasing ship size and traffic density there is currently a growing interest in the area of ship maneuvering especially in confined and shallow water. In spite of this the contributions from theoretical hydrodynamics to the field of ship maneuvering have been limited as pointed out by the

Maneuverability Committee of the 14th International Towing Tank Conference held in 1975. Following the recommendation of this conference, it was decided to analyse the problem of a general ellipsoid maneuvering in the proximity of a general disturbance (wall, free-surface, body, propeller, etc.). As is customary in maneuvering studies, the effect of compressibility, viscosity and Froude number will be neglected and the free surface will be replaced by the double-model approximation.

The determination of the hydrodynamical forces and moments acting on a maneuvering ship in a bounded or disturbed fluid medium is indeed a hard task. The problem may be simplified by using a tri-axial ellipsoid as an approximation to the ship hull taking advantage of some analytic tools which are particularly tailored for ellipsoidal shapes. The idea of replacing a ship hull by an "equivalent ellipsoid" was first suggested by Weinblum (1936) and later used by Eggers (1960), Wu and Chwang (1974), and Miloh and Landweber (1977).

The present paper considers the most general time-dependent motion of a tri-axial ellipsoid with six degrees of freedom in a disturbed fluid medium. The fluid is assumed to be inviscid and incompressible and the effect of fluid separation is ignored. The mathematical formulation employs the classical theory of ellipsoid harmonics (Hobson 1955) together with some recent contributions to potential flow theory (Miloh 1973, 1974). These theories are particularly appealing in view of the recent progress in computerizing the numerical computation of the various Lamé functions (Walter 1971).

The concept of "force and moment influence functions" is introduced to calculate the forces and moments experienced by a maneuvering ellipsoid in the proximity of an exterior disturbance which may be represented by an arbitrary distribution of singularities. First general expressions for the forces and moments acting on a maneuvering ellipsoid in the presence of an exterior

moving source with a time dependent output are derived. It is then shown how these "force and moment influence functions" may be used to determine the hydrodynamical forces and moments acting on an ellipsoid moving in the proximity of a general prescribed system of singularities representing, wall, free-surface, solid bodies, etc. This concept is somewhat analogous to the use of the Green's function of a unit source to determine the velocity potential of an arbitrary distribution of singularities by formulating an integral equation.

The general expression for the forces and moments acting on an ellipsoid in the proximity of an exterior source are obtained by applying the Lagally theorem to the image singularity system as described in Section 3 for the case of a stationary ellipsoid in the presence of exterior source, and in Section 4 for maneuvering ellipsoid in unbounded medium. The direct application of the Lagally theorem to determine the hydrodynamical forces and moments acting on a maneuvering ellipsoid in unbounded medium is demonstrated in Sections 5 and 6, respectively. Finally, expressions for the forces and moments experienced by an ellipsoid moving with six degrees of freedom in the proximity of an exterior unsteady source with arbitrary spacial motion are derived in Sections 7 and 8. It is also shown how these "influence functions" may be used to obtain closed-form expressions for the hydrodynamical forces and moments acting on the ellipsoid in the proximity of a general prescribed singularity system representing fluid or solid boundaries.

In Section 9 the general expressions thus derived are demonstrated to calculate the force experienced by the ellipsoid in a relatively simple case, namely an ellipsoid translating in the direction of its major axis in the presence of an exterior sink with time dependent output lying on the same axis. The results of this example may be found useful in analysing propeller - hull interaction, in accordance with Dickmann's isolated-sink model of the propeller. Other parameters such as potential wake fraction, thrust deduction coefficient and propeller induced vibratory force, are also briefly discussed.

2. MATHEMATICAL BACKGROUND

In order to solve the Laplace equation in a domain bounded (internally or externally) by a tri-axial ellipsoidal surface, it is convenient to formulate the problem in ellipsoidal coordinates. The ellipsoidal (confocal) coordinates are defined by the solutions of the following cubic equation in χ ;

$$\frac{x_1^2}{\chi^2} + \frac{x_2^2}{\chi^2 - h^2} + \frac{x_3^2}{\chi^2 - k^2} = 1 \quad (1)$$

for a fixed value of the Cartesian coordinates (x_1, x_2, x_3) . Both h and k are related to the three semi-axes of the ellipsoid i.e. a_1, a_2 , and a_3 in the x_1, x_2 and x_3 directions respectively by

$$k^2 = a_1^2 - a_3^2 ; \quad h^2 = a_1^2 - a_2^2 \quad (2)$$

such that $a_1 > a_2 > a_3$

The three roots of (1), here denoted by ρ, μ and ν , are so chosen that

$$\infty > \rho^2 \geq k^2 ; \quad k^2 \geq \mu^2 \geq h^2 ; \quad h^2 > \nu^2 \geq 0 \quad (3)$$

The three surfaces $\rho = \text{constant}$ (ellipsoids), $\mu = \text{constant}$ (hyperboloids of one sheet) and $\nu = \text{constant}$ (hyperboloids of two sheets) form a spacial triply orthogonal coordinate system. In such a coordinate system the canonical ellipsoid with the semi-axes (a_1, a_2, a_3) is simply given by $\rho = a_1$.

The transformation between the ellipsoidal and the Cartesian coordinates is given in Hobson (1955, p. 454)

$$x_1^2 = \frac{\rho^2 \mu^2 v^2}{h^2 k^2} \quad (4)$$

$$x_2^2 = \frac{(\rho^2 - h^2) (\mu^2 - h^2) (h^2 - v^2)}{h^2 (k^2 - h^2)} \quad (5)$$

$$x_3^2 = \frac{(\rho^2 - k^2) (k^2 - \mu^2) (k^2 - v^2)}{k^2 (k^2 - h^2)} \quad (6)$$

The three linearizing factors of the above transformation in the ρ , μ and v directions are here denoted by \bar{h}_ρ , \bar{h}_μ , and \bar{h}_v respectively and are given by

$$\bar{h}_\rho^2 = \frac{(\rho^2 - \mu^2) (\rho^2 - v^2)}{(\rho^2 - h^2) (\rho^2 - k^2)} \quad (7)$$

$$\bar{h}_\mu^2 = \frac{(\rho^2 - \mu^2) (\mu^2 - v^2)}{(\mu^2 - h^2) (k^2 - \mu^2)} \quad (8)$$

$$\bar{h}_v^2 = \frac{(\rho^2 - v^2) (\mu^2 - v^2)}{(h^2 - v^2) (k^2 - v^2)} \quad (9)$$

A normal (separable) solution of the Laplace equation at the point $M(\rho, \mu, v)$ which is regular at the origin may be written as

$$P_n^m(M) = P_n^m(\rho, \mu, v) = E_n^m(\rho) E_n^m(\mu) E_n^m(v) \quad (10)$$

where E_n^m denotes the Lamé function of the first kind of order n and degree m . Here both n and m are positive integers satisfying $2n + 1 \geq m \geq 1$.

Similarly a normal solution of the Laplace equation at $M(\rho, \mu, \nu)$ which is regular at infinity is

$$Q_n^m(M) = Q_n^m(\rho, \mu, \nu) = F_n^m(\rho) E_n^m(\mu) E_n^m(\nu) \quad (11)$$

where F_n^m denotes the Lamé function of the second kind which is related to E_n^m by

$$F_n^m(\rho) = I_n^m(\rho) E_n^m(\rho) \quad (12)$$

where, following Hobson (1955, p. 472)

$$I_n^m(\rho) = (2n+1) \int_0^\infty \frac{d\rho}{[E_n^m(\rho)]^2 \sqrt{(\rho^2 - h^2)(\rho^2 - k^2)}} \quad (13)$$

Both the interior $P_n^m(M)$ and the exterior $Q_n^m(M)$ ellipsoidal harmonics are linearly independent and form a complete set (Whittaker and Watson 1927, p. 560).

The orthogonality relations of the Lamé functions of the first kind may be expressed in the following form (Miloh 1973)

$$\int_S \frac{E_n^m(\mu) E_n^m(\nu) E_{n'}^{m'}(\mu) E_{n'}^{m'}(\nu)}{\sqrt{(a_1^2 - \mu^2)(a_1^2 - \nu^2)}} dS = 0 \quad (14)$$

for $n \neq n'$ or $m \neq m'$ where the integration is carried over the surface S of the canonical ellipsoid $\rho = a_1$. For the particular case where $n = n'$ and $m = m'$ we define

$$\gamma_n^m = \gamma_n^m(a_1, a_2, a_3) = \int_S \frac{[E_n^m(\mu) E_n^m(\nu)]^2}{\sqrt{(a_1^2 - \mu^2)(a_1^2 - \nu^2)}} dS \quad (15)$$

where it can be shown that in addition to dependence on the three semi-axes γ_n^m is a function of the transcendental number π .

In solving harmonic boundary value problems involving ellipsoidal surfaces, it is advantageous to have an expression for basic potential function which is the inverse of the distance between the two spacial points $M(\rho, \mu, \nu)$ and $M'(\rho', \mu', \nu')$ such that $\rho' > \rho$. Such an expression has been recently derived by Miloh (1973) in which the inverse value of the distance R between M and M' is given in terms of a double summation of ellipsoidal harmonics,

$$\frac{1}{R(M, M')} = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{4\pi}{(2n+1) \gamma_n^m} Q_n^m(M') P_n^m(M) \quad (16)$$

The above relation, which is essential in any formulation of the problem as an integral equation, serves as the basis for the following analysis.

3. THE IMAGE SINGULARITIES SYSTEM OF EXTERIOR SOURCE

In this section we derive an expression for the image singularities system over the surface of the ellipsoid which will preserve its shape in the presence of an exterior source. The velocity potential induced at M' by a unit sink located at M in an unbounded medium is given by (16). Introduction of an ellipsoidal boundary given by $\rho = a_1$ (such that $\rho' > a_1$) into the medium, disturbs the flow and the modified velocity potential of the source ϕ_s , may be written as

$$\phi_s(M, M') = - \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{4\pi}{(2n+1) \gamma_n^m} Q_n^m(M') [P_n^m(M) - C_n^m Q_n^m(M)] \quad (17)$$

where C_n^m are coefficients to be determined. Applying the boundary conditions of the Neumann type, i.e. that the normal derivative of ϕ_s vanishes on $\rho = a_1$ yields

$$C_n^m = C_n^m(a_1) = \frac{E_n^m(a_1)}{F_n^m(a_1)} \quad ; \quad D_n^m = D_n^m(a_1) = \frac{C_n^m}{(2n+1) \gamma_n^m} \quad (18)$$

where the dot denotes differentiation with respect to the argument, and D_n^m is defined here for later use.

It has been shown that an arbitrary exterior ellipsoidal harmonic may be represented by its "ultimate" image singularities system consisting of a continuous distribution of sources or normal doublets over the "fundamental ellipsoid" given by $\rho = k$ (Miloh 1974). Alternatively an exterior ellipsoidal harmonic may be also represented by its surface image singularities system which consists of surface distribution of sources over the canonical ellipsoid $\rho = a_1$ given by (Miloh 1973),

$$Q_n^m(M') = \frac{2n+1}{4\pi E_n^m(a_1)} \int_S \frac{E_n^m(\mu) E_n^m(\nu)}{\sqrt{(a_1^2 - \mu^2)(a_1^2 - \nu^2)}} \frac{dS}{R(M_e, M')} \quad (19)$$

where M_e denotes a point on the canonical ellipsoid i.e. $M_e(a_1, \mu, \nu)$.

Combining (16) and (19) renders the following expressions for the surface source distribution,

$$\sigma_s (M_e, M') = - \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{C_n^m Q_n^m (M')}{\gamma_n^m E_n^m (a_1)} \cdot \frac{E_n^m (\mu) E_n^m (\nu)}{\sqrt{(a_1^2 - \mu^2) (a_1^2 - \nu^2)}} \quad (20)$$

which is the desired expression for the induced source distribution at $M_e (a_1, \mu, \nu)$ resulting from an exterior source of unit strength lying at $M' (\rho', \mu', \nu')$.

4. THE IMAGE SINGULARITIES SYSTEM OF GENERAL MOTION IN UNBOUNDED MEDIUM

Consider a tri-axial ellipsoid moving with six degrees of freedom in an unbounded fluid medium otherwise at rest. The general motion consists of a translatory velocity vector $\vec{V}(V_1, V_2, V_3)$ where V_1, V_2 and V_3 denote the velocity components in the x_1, x_2 , and x_3 directions respectively. In addition, the ellipsoid is rotating about its major axes with angular velocity $\vec{\Omega}(\Omega_1, \Omega_2, \Omega_3)$ where again Ω_1, Ω_2 and Ω_3 denote the components of the angular velocity vector about x_1, x_2 and x_3 respectively. The six velocity components are assumed to be time dependent for the sake of generality. It is also understood that the Cartesian coordinate system (x_1, x_2, x_3) is rigidly attached to the ellipsoid.

The solution for the exterior velocity potential Φ_0 induced by such a motion at a point M_e on the ellipsoid may be found by solving six Neumann problems (Miloh 1973) which render,

$$\Phi_0 (M_e) = \sum_i [V_i \alpha_i (M_e) + \Omega_i \beta_i (M_e)] \quad i = 1, 2, 3 \quad (21)$$

The six coefficients $\alpha_i (M_e)$ and $\beta_i (M_e)$ are given by

$$\alpha_i (M_e) = KC_1^i (|a_{i+1}^2 - a_{i+2}^2|)^{1/2} Q_1^i (M_e) ;$$

$$\beta_i (M_e) = KC_2^{6-i} \frac{(|a_{i+1}^2 - a_{i+2}^2|)^{1/2}}{a_{i+1}^2 + a_{i+2}^2} Q_2^{6-i} (M_e)$$

(22)

where

$$1/K = [(a_1^2 - a_2^2) (a_2^2 - a_3^2) (a_1^2 - a_3^2)]^{1/2} = hk \sqrt{k^2 - h^2}$$

and defining $a_{3+i} = a_i$. The various Lamé functions appearing in (22) are defined by (Hobson 1955, p. 473).

$$E_1^1(\rho) = \rho ; \quad E_1^2(\rho) = \sqrt{(\rho^2 - h^2)} ; \quad E_1^3(\rho) = \sqrt{(\rho^2 - k^2)}$$

$$E_2^3(\rho) = \rho \sqrt{(\rho^2 - h^2)} ; \quad E_2^4(\rho) = \rho \sqrt{(\rho^2 - k^2)} ; \quad E_2^5(\rho) = \sqrt{(\rho^2 - h^2) (\rho^2 - k^2)}$$

implying that

$$P_1^1(M) = hkx_1 ; \quad P_1^2(M) = hx_2 \sqrt{k^2 - h^2} ; \quad P_1^3(M) = kx_3 \sqrt{k^2 - h^2}$$

$$P_2^3(M) = h^2 kx_1 x_2 \sqrt{k^2 - h^2} ; \quad P_2^4(M) = hk^2 x_2 x_3 \sqrt{k^2 - h^2} ; \quad P_2^5(M) = hk(k^2 - h^2) x_2 x_3$$

(23)

The various exterior harmonics Q_n^m appearing in (22) are related to the Lamé functions of the first kind listed in (23) in the manner given by (11), (12), and (13).

The final expression for the surface image source distribution σ_0 induced by the general motion at points M_e is obtained by substituting (19) in (21) which yields

$$\sigma_o (M_e) = - \frac{1}{\sqrt{a_1^2 - \mu^2} (a_1^2 - v^2)} \left\{ \sum_{i=1}^3 \frac{3V_i \alpha_i (M_e)}{4\pi a_1 F_1^i (a_1)} + \frac{5a_i \Omega_i \beta_i (M_e)}{3V F_2^{6-i} (a_1)} \right\} \quad (24)$$

where Ψ denotes the volume of the ellipsoid.

5. HYDRODYNAMICAL FORCES FOR MANEUVERING ELLIPSOID IN INFINITE MEDIUM

The hydrodynamical forces acting on a maneuvering Rankine body may be calculated by applying the Lagally (1922) theorem, once the image singularity system which generates the body has been determined. The powerful theorem of Lagally, originally given for steady flow conditions, has been extended by Cummins (1953) and by Landweber and Yih (1956) to cases of unsteady flows.

The desired expression for the Lagally force $\vec{T}(T_1, T_2, T_3)$ experienced by a Rankine body which is generated by a continuous surface source distribution is given by (Landweber and Yih 1956).

$$T_i = \rho_F \left[\Psi \frac{dv_i}{dt^*} - 4\pi \frac{d}{dt^*} \int_S \sigma x_i dS - 4\pi \int_S \sigma v_i dS \right] \quad (25)$$

where ρ_F denotes the mass density of the fluid, Ψ is the volume of the body, $\vec{V}_c(v_1, v_2, v_3)$ is the velocity of the body-volume centroid and $\vec{r}(x_1, x_2, x_3)$ denotes the position radius vector of a point lying on the body surface. The image system of sources of the exterior flow field is here denoted by $\sigma(\vec{r}, t)$ and $\vec{v}(v_1, v_2, v_3)$ is the velocity vector induced at the location of the singularity due to all external flow producing mechanisms, hence neglecting contributions from

all internal singularities. Time is represented by t and the symbol $\frac{d}{dt^*}$ denotes absolute time derivatives which are related to the relative time derivative $\frac{d}{dt}$ via

$$\frac{dq_i}{dt^*} = \frac{dq_i}{dt} + e_{ijk} \Omega_j q_k \quad i, j, k = 1, 2, 3 \quad (26)$$

for some arbitrary vector $\vec{q}(q_1, q_2, q_3)$ where e_{ijk} denotes the permutation tensor.

For the case where the ellipsoid is moving in an unbounded medium otherwise at rest there are no flow-producing mechanisms in the region exterior to the body and therefore $\vec{v} = 0$. Substituting this value in (26) and denoting the Lagally force for the unbounded case by $\vec{T}_0(T_{01}, T_{02}, T_{03})$, one obtains from (25)

$$T_{0i} = \rho_F \frac{d}{dt^*} [(\Psi v_i - 4\pi \int_S \sigma_o x_i dS)] \quad (27)$$

where $\sigma_o(\vec{r}, t)$ is given by (24).

An expression for the components x_i of the radius vector is obtained by substituting (22) and (23) in (4), (5) and (6) which yields

$$x_i = \frac{a_i \alpha_i(M)}{C_1^i F_1^i(a_1)} \quad (28)$$

Introducing (28) into (27) and using the orthogonality conditions of the Lamé functions given in (15), renders

$$\int_S \sigma_o x_i dS = -v_i C_1^i \quad (29)$$

where in deriving (29) we used the numerical values of the three coefficients γ_1^i ,

$$\gamma_1^i = \frac{4\pi}{3K^2(a_{i+1}^2 - a_{i+2}^2)} \quad (30)$$

as computed by the substitution of (23) into (15) and performing the integration over $\rho = a_1$.

The final result for the Lagally force for maneuvering ellipsoid in unbounded medium is obtained after operating with the absolute time derivative on (27) which yields,

$$T_{oi} = \rho_F \Psi \left\{ \left(1 + \frac{3C_1^i}{K}\right) \frac{dv_i}{dt} - e_{ijk} \left(1 + \frac{3C_1^j}{K}\right) v_j \Omega_k \right\} \quad (31)$$

The above relation may be further simplified by recalling the Taylor (1928) added mass theorem which renders the following expressions for the three translatory added mass coefficients A_{ii} corresponding to pure translation in the x_i directions

$$\Psi(1 + A_{ii}) = 4\pi \int_S \sigma_o x_i dS \quad (32)$$

which, because of (29), yields

$$A_{ii} = - \left(1 + \frac{4\pi C_1^i}{\Psi}\right) \quad (33)$$

Substitution of (33) into (31) enables us to express the latter in the more familiar form

$$T_{oi} = -\rho_F V \left[A_{ii} \frac{dV_i}{dt} - e_{ijk} A_{jj} V_j \Omega_k \right] \quad (34)$$

The above relation is also valid for the more general class of maneuvering bodies with three orthogonal planes of symmetry (Kochin et al., 1964, p. 401) in an undisturbed medium. The classical derivation of (34) is by employing the method of Kirchhoff-Lagrange (Lamb 1932, p. 188) which is based on energy conservation. Here we rederive the same relations for ellipsoidal bodies using a different approach, i.e., by applying the Lagally and the Taylor theorems which are based on the momentum conservation principle. It should be also mentioned that for ellipsoidal bodies there exist closed-form expressions for the three translatory added-mass coefficients in terms of the tabulated Legendre incomplete elliptic integrals (Miloh 1973, 1974).

6. HYDRODYNAMICAL MOMENTS FOR MANEUVERING ELLIPSOID IN INFINITE MEDIUM

In order to express the Lagally moment experienced by a Rankine body in terms of its image system of singularities it was found convenient to decompose the exterior velocity potential Φ into two parts; one which is due to the motion of the body alone which is again decomposed into six velocity potentials in the manner of Kirchhoff, and a second part Φ_s which is due to the motion of the external boundaries and other flow-producing mechanisms when the body is at rest. The total exterior velocity potential is then written as

$$\Phi(M) = \sum_{i=1}^3 [V_i \phi_i(M) + \Omega_i \phi_{3+i}(M)] + \phi_s(M) \quad (35)$$

where the various velocity potentials in (35) satisfy the following kinematic boundary conditions on the body surface.

$$\frac{\partial \phi_i}{\partial \tilde{n}} = \tilde{n}_i \quad ; \quad \frac{\partial \phi_{3+i}}{\partial \tilde{n}} = e_{ijk} \tilde{x}_j \tilde{n}_k \quad ; \quad \frac{\partial \phi_s}{\partial \tilde{n}} = 0 \quad (36)$$

Here $\vec{\tilde{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ denotes the unit normal vector in the outward direction of the surface.

The Lagally expressions for the hydrodynamical moment $\vec{M} = (M_1, M_2, M_3)$ are somewhat more complicated than the Lagally-force expressions given in the previous section and are given by (Landweber and Yih 1956).

$$M_i = \rho_F V \left[e_{ijk} \tilde{x}_j \frac{dV_k}{dt} \right] - \frac{d}{dt^*} \sum_{j=1}^3 (\Omega_j A_{3+i, 3+j}) - 4\pi \rho_F \frac{d}{dt^*} \quad (37)$$

$$\int_S \left\{ \sigma_{3+i} \left[\sum_{j=i}^3 \tilde{x}_j v_j - \phi_s \right] + \sigma_s \phi_{3+i} \right\} dS - 4\pi \rho_F e_{ijk} \int_S [\sigma \tilde{x}_j (v_k - V_k)] dS$$

where $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ denote the position vector of the volume centroid of the body and $A_{3+i, 3+j}$ are the rotational added mass coefficients of the body. The surface source distribution here denoted by σ_{3+i} and by σ_s represent the image singularity system of the exterior velocity potentials ϕ_{3+i} and ϕ_s respectively. The sum of all surface source images corresponding to both the motion of the body and other boundaries and flow-producing mechanisms is represented by σ .

Since the ellipsoid has three mutual perpendicular planes of symmetry we have only three rotational added mass coefficients of the type $A_{3+i, 3+i}$, which corresponds to pure rotation about the \tilde{x}_i axis, and in addition $\tilde{x}_i = 0$. Further-

more, for the present case where the ellipsoid is maneuvering in an unbounded fluid medium without exterior flow-producing mechanisms, equation (37) may be simplified to yield the following expressions for the undisturbed Lagally moment $\vec{M}_o(M_{o1}, M_{o2}, M_{o3})$

$$M_{oi} = -\rho_F \nabla \frac{d}{dt^*} (\Omega_i A_{3+i, 3+i}) + 4\pi\rho_F e_{ijk} \int_S \sigma_o x_j V_k dS \quad (38)$$

Substituting (29) and (33) in (38) and using (26) for the absolute time derivative yields the final expressions for \vec{M}_o ,

$$M_{oi} = -\rho_F \nabla [A_{3+i, 3+i} \frac{d\Omega_i}{dt} - B_{3+i} \Omega_{i+1} \Omega_{i+2} - B_{-3+i} V_{i+1} V_{i+2}] \quad (39)$$

where we define for $i = 1, 2, 3$.

$$V_{3+i} = V_i, \Omega_{3+i} = \Omega_i; A_{-3+i, -3+i} = A_{ii}; A_{6+i, 6+i} = A_{3+i, 3+i} \quad (41)$$

$$B_i = A_{i+1, i+1} - A_{i+2, i+2}$$

Again the novelty in (39) is in the approach rather than in the result which was also given for example in Kochin et al. (1964, p. 401). Here we re-derive the classical expression for the hydrodynamical moment acting on a maneuvering body with three orthogonal planes of symmetry in terms of its added masses, using the momentum approach which involves integration of the pressure over the surface rather than employing the energy (Kirchhoff's) method (Lamb 1932, p. 168). It should also be mentioned that the Lagally theorem is more general than the Kirchhoff-Lagrange method in the sense that it enables us to compute the forces and moments on Rankine bodies in cases where the latter method can not be

applied. (Landweber and Yih 1956)

The three pure-rotational added mass coefficients $A_{3+i, 3+i}$ appear explicitly in the Lagally moment expression since there exists no generalization of the Taylor added mass theorem for pure rotation which relates the added masses to the corresponding image singularity system. However, by applying the kinetic-energy theorem (Lamb 1932, p. 153-155) one can express the rotational added masses of an ellipsoid in terms of the coefficients C_2^i defined in (18) (Miloh 1973) as

$$A_{3+i, 3+i} = - \frac{(a_{i+1}^2 - a_{i+2}^2)^2}{5(a_{i+1}^2 + a_{i+2}^2)} \left[1 + \frac{20\pi C_2^{6-i}}{3\mathcal{W}(a_{i+1}^2 + a_{i+2}^2)} \right] \quad (42)$$

The coefficients C_2^i in turn may also be expressed in terms of the tabulated incomplete Legendre elliptic functions (Miloh 1973).

7. HYDRODYNAMICAL FORCES IN DISTURBED OR BOUNDED FLUID DOMAIN

In this section we derive expressions for the Lagally forces acting on a maneuvering ellipsoid in the presence of exterior disturbance resulting from the proximity of other bodies or boundaries as well as isolated flow-producing singularities. The external boundaries will be represented by a surface distribution of sources which may be continuous or discrete. We will consider only sources as a model for treating exterior flow-producing singularities where it is understood that higher order singularities may also be handled by the same method by a proper differentiation. It was found instructive to derive the first expression for the Lagally forces resulting from the presence of an exterior source, before dealing with the more general problem concerning arbitrary fluid disturbances.

Consider an isolated source of strength $\delta(t)$ lying at $M'(\rho', \mu', \nu')$ such that $\rho' > a_1$, which implies that the source lies in the domain exterior to the canonical ellipsoid. The source may translate in an arbitrary manner such that the instantaneous location of the source in a Cartesian coordinate system attached to the ellipsoid is $M'(\vec{r}')$ where $\vec{r}'(x'_1, x'_2, x'_3)$ is a time dependent radius vector.

It is convenient to decompose the Lagally force $\vec{T}(T_1, T_2, T_3)$ acting on a maneuvering ellipsoid in the presence of an exterior disturbance into three terms

$$T_i = T_{oi} + T_{pi} + T_{qi} \quad (43)$$

where as before $\vec{T}_o(T_{o1}, T_{o2}, T_{o3})$ denotes the force acting on the ellipsoid when the exterior disturbance is not present, $\vec{T}_p(T_{p1}, T_{p2}, T_{p3})$ is the force acting on a stationary ellipsoid where the exterior disturbances is moving and $\vec{T}_q(T_{q1}, T_{q2}, T_{q3})$ is the contribution from the interaction between the motions of both the ellipsoid and the exterior disturbances.

Applying the general Lagally formula (25) for the case where the exterior disturbance is an isolated source, yields the following expressions for \vec{T}_p and \vec{T}_q :

$$T_{pi} = 4\pi\rho_F \frac{d}{dt} \int_S \sigma_s x_i dS - 4\pi\rho_F \int_S \sigma_s v_i dS \quad (44)$$

and

$$T_{qi} = 4\pi\rho_F e_{ijk} \Omega_j \int_S \sigma_s x_k dS - 4\pi\rho_F \int_S \sigma_o v_i dS \quad (45)$$

where $\sigma_s(M_e, M')$ is the image source distribution induced at M_e due to the exterior source lying at M' and is given by (20) multiplied by the source strength δ . The image distribution for maneuvering ellipsoid in unbounded medium $\sigma_o(M_e)$ appearing in (45) is given by (24). The velocity induced at M_e due to the presence of the source at M' is denoted by $\vec{v}(v_1, v_2, v_3)$ and may be expressed as,

$$v_i(M_e, M') = -\delta \frac{\partial}{\partial x_i} \left[\frac{1}{R(M, M')} \right] \Bigg|_{M=M_e} = \delta \frac{\partial}{\partial x_i} \left[\frac{1}{R(M_e, M')} \right] \quad (46)$$

or, after substituting (16) into (46)

$$v_i(M_e, M') = 4\pi \delta \frac{\partial}{\partial x_i} \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{P_n^m(M_e)}{(2n+1)\gamma_n^m} Q_n^m(M') \quad (47)$$

Let us turn now to the computation of the various integrals occurring in (44) and (45). Substitution of (28) and (42) into the first integrals in the right hand side of (44) yields after applying the orthogonality properties of the Lamé functions

$$\int_S \sigma_s x_i dS = -\delta \alpha_i(M') \quad (48)$$

Similarly, substitution of (28) and (47) renders for the second integral in (44)

$$\int_S \sigma_{s_i} v_i dS = - 2\pi\delta^2 \frac{\partial}{\partial x_i} \sum_{n=0}^{2n+1} \sum_{m=1} D_n^m [Q_n^m (M')]^2 \quad (49)$$

where D_n^m has been defined in (18).

The above relations yield the final form for the disturbance Lagally force \vec{T}_p ,

$$T_{pi} = 4\pi\rho_F \frac{d}{dt} [\delta\alpha_i (M')] - 8\pi^2\rho_F\delta^2 \frac{\partial}{\partial x_i} \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m [Q_n^m (M')]^2 \quad (50)$$

The interaction force \vec{T}_q will be considered next. Substituting (24) and (47) into (45) yields

$$\int_S \sigma_{o_i} v_i dS = - \delta \frac{\partial}{\partial x_i} \sum_{i=1}^3 [v_i \alpha_i (M') + \Omega_i \beta_i (M')] \quad (51)$$

Finally, applying (48) we get the following expression for the interaction force

$$T_{qi} = 4\pi\rho_F \delta e_{ijk} \Omega_j \alpha_k (M') + 4\pi\rho_F \delta \frac{\partial}{\partial x_i} \sum_{i=1}^3 [v_i \alpha_i (M') + \Omega_i \beta_i (M')] \quad (52)$$

The general expression for the Lagally force acting on a maneuvering ellipsoid in the presence of exterior source where both its strength and location are time dependent, is found by summing the different contributions given by (34) (50) and (52) namely

$$T_i/4\pi\rho_F = \frac{d}{dt} [\delta\alpha_i (M') - \nabla A_{ii} v_i] + e_{ijk} \Omega_j [\delta\alpha_k (M') - \nabla A_{kk} v_k] \quad (53)$$

$$+ \frac{\partial}{\partial x'_i} \left\{ \sum_{j=1}^3 [v_j \alpha_j (M') + \Omega_j \beta_j (M')] - 2\pi \delta \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m [Q_n^m (M')]^2 \right\}$$

Once the expression for the Lagally force acting on a maneuvering ellipsoid by an exterior source has been determined it is easy to calculate the corresponding force due to an arbitrary exterior disturbance. Let us assume that the exterior disturbance is represented by a continuous or discrete source distribution $\delta' (M', t)$ over the surface S' which may be finite or infinite such that the velocity potential induced at points on the canonical ellipsoid is given by

$$\Phi (M_e) = - \int_{S'} \delta' (M', t) \frac{dS'}{R(M', M_e)} \quad (54)$$

Here S' is the set of all points M' . For the case of discrete singularities the integral in (54) may be replaced by a summation over all singularities. For the present case equation (53) may be generalized to yield

$$T_i/4\pi\rho_F = \frac{d}{dt} \left[\int_{S'} \delta' \alpha_i (M') dS' - \nabla A_{ii} v_i \right] + e_{ijk} \Omega_j \left[\int_{S'} \delta' \alpha_k (M') dS' - \nabla A_{kk} v_k \right] \quad (55)$$

$$+ \int_{S'} \delta' \sum_{j=1}^3 \frac{\partial}{\partial x'_i} [v_j \alpha_j (M') + \Omega_j \beta_j (M')] dS'$$

$$- 2\pi \int_{S'} \int_{S''} \delta' \delta'' \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m Q_n^m (M') \frac{\partial}{\partial x'_i} [Q_n^m (M')] dS' dS''$$

where $S'' = S'$ and $M'' = M'$.

On the assumption that the singularities representation of the exterior disturbance δ' is known the above expression enables us to calculate directly the force on a maneuvering ellipsoid in the proximity of an arbitrary disturbance. The first term in (55) represents the effect of the unsteadiness of the problem, the next two terms the additional force caused by the motion of the ellipsoid. The last term is the interaction force which is also the force on a stationary ellipsoid due to stationary external disturbance. It arises from the fact that the singularities of the external disturbance have an image within the ellipsoid. The interaction force is usually small in comparison with the terms arising from the motion of the ellipsoid.

8. HYDRODYNAMICAL MOMENT IN DISTURBED OR BOUNDED FLUID DOMAIN

In a similar manner we start with the derivation of the Lagally moment acting on a maneuvering ellipsoid in the presence of an exterior source. Again we decompose the Lagally moment $\vec{M}(M_1, M_2, M_3)$ into three terms.

$$M_i = M_{oi} + M_{pi} + M_{qi} \quad i = 1, 2, 3 \quad (56)$$

where $\vec{M}_o (M_{o1}, M_{o2}, M_{o3})$ denotes the moment in undisturbed medium, $\vec{M}_p (M_{p1}, M_{p2}, M_{p3})$ is the moment acting on a stationary ellipsoid and $\vec{M}_q (M_{q1}, M_{q2}, M_{q3})$ is the additional moment caused by the motion of the ellipsoid in the proximity of the disturbance.

The undisturbed Lagally moment \vec{M}_o has been derived in (39) and for \vec{M}_p and \vec{M}_q we may write the following (37),

$$M_{pi} = - 4\pi\rho_F e_{ijk} \int_S \sigma_s x_j v_k dS \quad (57)$$

and

$$M_{qi} = -4\pi\rho_F \frac{d}{dt} \int_S (\sigma_s \phi_{3+i} - \phi_{3+i} \sigma_s) dS - 4\pi\rho_F e_{ijk} \int_S x_j (\sigma_s v_k - \sigma_s v_k) dS \quad (58)$$

The value of \vec{M}_p will be considered first. The following relation may be easily verified.

$$e_{ijk} x_j \frac{\partial}{\partial x_k} \left[\frac{1}{R(M, M')} \right] = e_{ijk} x'_j \frac{\partial}{\partial x'_k} \left[\frac{1}{R(M, M')} \right] \quad (59)$$

Where here $M(x_1, x_2, x_3)$ and $M'(x'_1, x'_2, x'_3)$. Substituting (46), (47) and (59) into (57) one gets

$$M_{pi} = - 4\pi\rho_F e_{ijk} x'_j \int_S \sigma_s v_k dS \quad (60)$$

Which in view of (49) yields

$$M_{pi} = 8\pi^2 \rho_F \delta^2 e_{ijk} x'_j \frac{\partial}{\partial x'_k} \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m [Q_n^m(M')]^2 \quad (61)$$

The interaction moment \vec{M}_q is considered next. From (21) and (24) we have the following expressions;

$$\phi_{3+i}(M_e) = \beta_i (M_e) \quad (62)$$

Substituting (64) and (66) into (58) yields the following expression for the interaction moment.

$$M_{qi}/4\pi\rho_F = \frac{d}{dt} [\delta\beta_i(M')] + \delta e_{ijk} \left\{ v_k \alpha_j(M') + \Omega_j \beta_k(M') + x_j' \frac{\partial}{\partial x_k'} \sum_{j=1}^3 [V_j \alpha_j(M') + \Omega_j \beta_j(M')] \right\} \quad (67)$$

The final expression for the Lagally moment acting on a maneuvering ellipsoid in the proximity of an exterior source is obtained by summing up the three different contributions to the moment given in (39), (61) and (67) which yields

$$\begin{aligned} M_i/4\pi\rho_F = & \frac{d}{dt} \left[-\frac{v}{4\pi} A_{3+i, 3+i} \Omega_i + \delta\beta_i(M') \right] + \frac{v}{4\pi} [B_{3+i} \Omega_{i+1} \Omega_{i+2} + B_{-3+i} v_{i+1} v_{i+2}] \\ & + \delta e_{ijk} [v_k \alpha_j(M') + \Omega_j \beta_k(M')] + \delta e_{ijk} x_j' \frac{\partial}{\partial x_k'} \left\{ \sum_{j=1}^3 [V_j \alpha_j(M') + \Omega_j \beta_j(M')] \right\} \\ & + 2\pi\delta \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m [Q_n^m(M')]^2 \quad (68) \end{aligned}$$

For the general case where the velocity potential induced by the exterior disturbance is given by (54) the expression for the Lagally moment becomes

$$M_i/4\pi\rho_F = \frac{d}{dt} \left[-\frac{v}{4\pi} A_{3+i, 3+i} \Omega_i + \int_{S'} \delta' \beta_i(M') dS' \right] + \frac{v}{4\pi} [B_{3+i} \Omega_{i+1} \Omega_{i+2} +$$

and

$$\sigma_{3+i}(M_e) = - \frac{5a_i \beta_i(M_e)}{3\sqrt{F_2}^{6-i} (a_1) \sqrt{(a_1^2 - \mu^2)(a_1^2 - \nu^2)}} \quad (63)$$

The disturbance velocity potential $\phi_s(M_e, M')$ and its image system of singularities induced at the surface of the ellipsoid $\sigma_s(M_e, M')$ are given by (17) and (20) respectively after multiplication by the source strength δ . Substituting (17), (20), (62) and (64) into the first integral in (58) yields

$$\int_S (\sigma_s \phi_{3+i} - \sigma_{3+i} \phi_s) dS = - \delta \beta_i(M') \quad (64)$$

The second integral on the right hand side of (59) may be also written as

$$e_{ijk} \int_S x_j (\sigma_o v_k - \sigma_s v_k) dS = e_{ijk} [x'_j \int_S \sigma_o v_k dS - v_k \int_S \sigma_s x_j dS] \quad (65)$$

In the above form the relations given in (48) and (51) are readily used to give

$$e_{ijk} \int_S x_j (\sigma_o v_k - \sigma_s v_k) dS = - \delta e_{ijk} \left\{ v_k \alpha_j(M') + x'_j \frac{\partial}{\partial x'_k} \sum_{j=1}^3 [v_j \alpha_j(M') + \Omega_j \beta_j(M')] \right\} \quad (66)$$

$$\begin{aligned}
 & + B_{-3+i} V_{i+1} V_{i+2}] + e_{ijk} \int_{S'} \delta' [V_k \alpha_j (M') + \Omega_j \beta_k (M')] dS' \\
 & + e_{ijk} x_j^1 \int_{S'} \delta' \sum_{j=1}^3 \frac{\partial}{\partial x_k^1} [V_j \alpha_j (M') + \Omega_j \beta_j (M')] dS' \quad (69) \\
 & + 2\pi e_{ijk} x_j^1 \int_{S'} \int_{S''} \delta' \delta'' \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m Q_n^m (M'') \frac{\partial}{\partial x_k^1} [Q_n^m (M')] dS' dS''
 \end{aligned}$$

Thus the hydrodynamical moment acting on a maneuvering ellipsoid in a bounded or disturbed medium may be calculated directly from (69) once the source singularities representation of the exterior disturbance is specified. As before, for discrete source distribution the integral signs in (69) are replaced by a summation over all singularities. The derived expressions for the Lagally force and moment may be also modified to account for higher order singularities by a proper partial differentiation of the terms under the integral sign.

9. MODEL EXAMPLE: PROPELLER-HULL INTERACTION

As a demonstration of the general expressions thus given for the hydrodynamical forces and moments experienced by a maneuvering tri-axial ellipsoid, we consider the effect of placing a thrust-producing mechanism on the major axis of a self-propelled ellipsoid. In addition, we exclude rotational motion of the ellipsoid and consider only translation in the direction of the major axis. The aim is to determine the effect of such a thrust-producing system on the flow pattern about the hull, with special reference to the determination of basic parameters such as wake fraction, thrust deduction and propeller-hull exciting forces.

Following the classical work of Dickmann (1938) the hydrodynamical effect of a single propeller located at the ship's stern may be represented by a point sink in accordance with the actuator disk theory. The strength of the sink, lying at $x = x_p > a_1$; $y = z = 0$ (or $\rho = x_p > a_1$; $\mu = k$; $\nu = h$,) is denoted by $\delta(t) < 0$. The time dependent output of the sink may be also written as

$$\delta(t) = \bar{\delta} + \tilde{\delta} e^{ib\omega t} \quad (70)$$

where $\bar{\delta}$ denotes the mean (steady) output of the sink and $\tilde{\delta}$ is the amplitude of the harmonic time dependent variation of the sink output. The angular velocity of the propeller is denoted by ω and b is the number of blades.

The mean sink strength may also be expressed in terms of the thrust loading coefficient \bar{C}_{TH} , thus following Dickmann's analysis,

$$\bar{\delta} = - \frac{A\bar{V}}{4\pi} [(1 + \bar{C}_{TH})^{1/2} - 1] \quad (71)$$

where A is the propeller area taken as a disk excluding the hub, and the thrust loading coefficient is defined by

$$\bar{C}_{TH} = \frac{2\bar{T}_h}{\rho\bar{V}^2 A} \quad (72)$$

Here \bar{T}_h denotes the total mean thrust required to attain a constant cruising velocity \bar{V} . For a propeller operating in the wake behind a hull, a better model for the sink strength is believed to be the one suggested by Nowacki and Sharma (1971) in which equation (71) is augmented by

$$\bar{\delta} = - \frac{A\bar{V}}{4\pi} (1 - \bar{W}_e) [(1 + \bar{C}_{TH})^{1/2} - 1] \quad (73)$$

The mean effective wake fraction \bar{W}_e is defined as the ratio between the mean decremental velocity over the propeller disk in the presence of the propeller and the advance velocity. The presence of a propeller behind the hull will also alter the pressure field about the hull giving rise to an additional interaction force between the propeller and the hull. Because of that reason, the propeller must develop a larger thrust in the case of a self-propelled hull than the thrust needed when the hull is towed. The ratio between the propeller-hull interaction force and the total propeller thrust is the so called thrust deduction coefficient λ_p .

$$\lambda_p = \frac{T_1}{T_h} \quad (74)$$

where T_1 denotes the absolute value of the propeller induced deduction-thrust on the hull. In the present demonstration we wish to examine the relation between the thrust deduction and the effective wake fraction for ellipsoidal hulls.

Since both the hull and the propeller move with identical velocity $V(t)$ in the direction of the negative x_1 direction, the interaction force between the propeller and the hull is given by (53),

$$\frac{T_1}{4\pi\rho_F} = - \alpha_1(M') \frac{d\delta}{dt} + \frac{\Psi}{4\pi} A_{11} \frac{dV}{dt} - \delta V \frac{d\alpha_1(M')}{dx_1} + 2\pi\delta^2 \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m \frac{d}{dx_1} [Q_n^m(M')]^2$$

$$x_1 = x_p \quad (75)$$

The mean value, of (75) is,

$$\frac{\bar{T}_1}{4\pi\rho_F} = -\bar{\delta}V \frac{d\alpha_1(M')}{dx_1} + 2\pi\bar{\delta}^2 \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m \frac{d}{dx_1} [Q_n^m(M')]^2 \Big|_{x_1 = x_p} \quad (76)$$

which is also the steady component of the thrust deduction. For most practical purposes $V \gg \bar{\delta}$ and it is sufficient to consider only the first non-vanishing term in the double summation in (76) (assuming that the series converges uniformly). For $n=0$ there is no contribution from the double series since $D_0^1 = 0$ and the only non-vanishing term for $n=1$ corresponds to $m=1$. Thus equation (76) together with (18), (22) and (30) yields

$$\frac{\bar{T}_1}{4\pi\rho_F} \approx -\frac{\bar{\delta}C_1^1}{hk} \cdot \frac{d}{dx_1} Q_1^1(M') \left[V - \frac{\delta Q_1^1(M')}{hk} \right] \Big|_{x_1 = x_p} \quad (77)$$

where (Miloh 1974)

$$Q_1^1(M') = 3hkx_p \int_{x_p}^{\infty} \frac{d\rho}{\rho^2 \sqrt{(\rho^2 - h^2)(\rho^2 - k^2)}} = \frac{3x_p}{k} [F(\theta, \lambda) - E(\theta, \lambda)] \quad (78)$$

Here F and E denote the Legendre incomplete elliptic integrals of the first and second kind respectively (Byrd and Friedman 1971) with arguments given by,

$$\theta = \sin^{-1} \frac{k}{x_p} \quad ; \quad \lambda = \frac{h}{k} \quad (79)$$

It then follows that

$$\left. \frac{d}{dx_1} Q_1^1(M') \right|_{x_1=x_p} = \frac{3}{k} [F(\theta, \lambda) - E(\theta, \lambda)] - \frac{3hk}{x_p \sqrt{(x_p^2 - h^2)(x_p^2 - k^2)}} \quad (80)$$

The additional coefficient in (77) namely C_1^1 is defined in (33) in terms of the longitudinal added mass coefficient of the ellipsoid, A_{11} and may be also expressed, in a similar manner to (80) by,

$$\frac{1}{C_1^1} = \frac{3}{hk^2} [F(\theta^*, \lambda) - E(\theta^*, \lambda)] - \frac{3}{a_1 a_2 a_3} \quad (81)$$

where here $\theta^* = \sin^{-1} \frac{k}{a}$. It should be noted that the r.h.s. of (81) equals the r.h.s. of (80) divided by hk where x_p is replaced by a_1 . Thus, equation (77) is an analytic expression for the thrust deduction in terms of the propeller output and its location along the major axis and the hull geometry. The various functions in (77) are given in (78), (80) and (81) in terms of the tabulated Legendre elliptic integrals which are suitable forms for numerical computation.

In order to determine the magnitude of the effective wake fraction, we apply equations (17) and (23) which render

$$\bar{w}_e = \frac{C_1^1}{hk} \frac{dQ_1^1(M')}{dx_1} - 2\pi(\bar{\delta}/V) \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m \left. \frac{d}{dx_1} [Q_n^m(M')] \right|_{x_1=x_p} \quad (82)$$

Comparing (76) and (82) implies that the relation between the thrust deduction and the effective wake fraction is given by,

$$\bar{T}_1 = -4\pi\rho_F \bar{\delta} V \bar{w}_e \quad (83)$$

which may also be obtained directly by considering the Lagally force on the sink in the presence of the wake.

The non-dimensional representation of (83) is

$$\lambda_p = \frac{2\bar{w}_e (1-\bar{w}_e)}{1 + \sqrt{1 + C_{TH}}} \quad (84)$$

which is obtained by substituting (72), (73) and (74) in (83). The above relation between the effective wake fraction, the thrust deduction coefficient and thrust loading coefficient is exact within the limits of potential flow theory and Dickmann's model of the propeller. A similar relation, aside from the correction term $1 - \bar{w}_e$, has been given by Tsakonas and Jacobs (1960) in their study of prolate spheroid. Tsakonas and Jacobs derivation ignores the contribution of images in the body which are required to maintain its outline in the presence of the propeller singularity. It should be noted that we were able to show that the relation holds also in the case where the propeller images are taken into account, thus the relation between the thrust-deduction and the wake fraction has new meaning. It is true that for elongated slender bodies Kroukovsky (1956) was able to show that the effect of the propeller images is small, but on the other hand Martinek and Yeh (1956) pointed out that this effect is important and may in fact explain some of the discrepancies between theory and experiments especially for blunt bodies.

Equation (82) can be solved explicitly for the effective wake fraction when using the definition (73) for the sink strength, yielding,

$$\bar{W}_e = \frac{\left[\frac{C_1^1}{hk} \frac{d}{dx_1} Q_1^1 (M') + 4\pi hk A^* \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m Q_n^m (M') \frac{d}{dx_1} Q_n^m (M') \right]}{\left[1 + 4\pi hk A^* \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m Q_n^m (M') \frac{d}{dx_1} Q_n^m (M') \right]} \quad (85)$$

where $A^* = \frac{A}{4\pi hk} [\sqrt{1 + C_{TH}} - 1]$ is a non-dimensional parameter which is small compared to unity for lightly loaded propellers and for elongated hulls. Numerical experience with the double series in (85) shows that for $\rho' > a_1$ the series converges fast and it is sufficient to consider only the first term in the series which corresponds to $n=1$ and $m=1$. For $n=2$ there are only two non-zero coefficients corresponding to $m=1$ and $m=2$. Hence equation (85) can be approximated by

$$\bar{W}_e \approx \frac{C_1^1}{hk} \frac{d}{dx_1} Q_1^1 (M') \left\{ 1 + A^* Q_1^1 (M') \left[1 - \frac{C_1^1}{hk} \frac{d}{dx_1} Q_1^1 (M') \right] \right\} \quad (86)$$

where the coefficient C_1^1 , the ellipsoidal harmonic $Q_1^1(M')$ and its partial derivative with respect to x_1 evaluated at $x_1 = x_p$, are given by (78), (79) and (81).

For prolate spheroidal shapes, $a_2 = a_3$ or $h = k$ and equation (86) reduces to,

$$\bar{W}_e = \frac{\log \frac{x_p - h}{x_p + h} + \frac{2hx_p}{x_p^2 - h^2}}{\log \frac{a_1 - h}{a_1 + h} + \frac{2ha_1}{a_1^2 - h^2}} \left\{ 1 - \frac{3}{2} A^* \left[\frac{x_p}{h} \log \frac{x_p - h}{x_p + h} + 2 \right] \left[1 - \frac{\log \frac{x_p - h}{x_p + h} + \frac{2hx_p}{x_p^2 - h^2}}{\log \frac{a_1 - h}{a_1 + h} + \frac{2ha_1}{a_1^2 - h^2}} \right] \right\} \quad (87)$$

which can be directly obtained from (78), (80) and (81) by applying the limit $k \rightarrow h$.

When $\rho' \rightarrow a_1$ the term multiplying the parenthesis of (86) is unity implying that $\bar{W}_e(\rho' \rightarrow a_1) = 1$. Similarly for large values of ρ' we get $\bar{W}_e(\rho' \rightarrow \infty) = 0$. Hence the thrust deduction coefficient as given in (84), vanishes in both limits as $\rho' \rightarrow a_1$ and $\rho' \rightarrow \infty$ and has a local maximum of $\frac{1}{2} [1 + \sqrt{1 + C_{TH}}]^{-1}$ (0.25 for lightly loaded propellers) at $\bar{W}_e = 0.5$.

The time dependent component of the thrust deduction which induces the hull vibrations is found by substituting (70) in (75) and considering time-harmonic terms of blade frequency $b\omega$,

$$\begin{aligned} \frac{\tilde{T}_1}{4\pi\rho_F} = & -\alpha_1(M') \frac{d}{dt} (\tilde{\delta} e^{ib\omega t}) - \tilde{\delta} V e^{ib\omega t} \frac{d\alpha_1(M')}{dx_1} \\ & + 4\pi\tilde{\delta} \tilde{\delta} e^{ib\omega t} \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} D_n^m \frac{d}{dx_1} [Q_n^m(M')]^2 \end{aligned} \quad (88)$$

where only the real part has to be considered.

In deriving (88) it was also assumed that the ship longitudinal acceleration $\frac{dV}{dt}$ is negligible due to the large inertia of the ship. The resulting relations for the vibratory force, when considering only the first non-vanishing term in the infinite series of (88), may be expressed as,

$$\tilde{T}_1 = 4\pi\rho_F \Theta e^{i(b\omega t + \theta)} \quad (89)$$

where Θ and θ are the amplitude and the phase angle of the vibratory force respectively, which are given by

$$\Theta = -\tilde{\delta} \frac{C_1^1}{hk} \sqrt{\theta_1^2 + \theta_2^2}, \quad \theta = \text{tg}^{-1} (\theta_2/\theta_1) \quad (90)$$

and

$$\Theta_1 \equiv \frac{d}{dx_1} Q_1^1 (M') [V - 2\bar{\delta}Q_1^1(M')] \quad (91)$$

$$\Theta_2 = b\omega Q_1^1 (M') \quad (92)$$

For a spheroidal shape ($k \rightarrow h$) equation (90) reduces to

$$\Theta = \frac{2h\tilde{\delta}}{3 \left[\log \frac{a_1-h}{a_1+h} + \frac{2ha_1}{a_1-h^2} \right]} \sqrt{\Theta_1^2 + \Theta_2^2} \quad (93)$$

where,

$$\Theta_1 = \frac{3}{2h} \left[\log \frac{x_p-h}{x_p+h} + \frac{2hx_p}{x_p^2-h^2} \right] \left\{ V + 3\bar{\delta} \left[\frac{x_p}{h} \log \frac{x_p-h}{x_p+h} + 2 \right] \right\} \quad (94)$$

$$\Theta_2 = -\frac{3}{2} b\omega \left[\frac{x_p}{h} \log \frac{x_p-h}{x_p+h} + 2 \right] \quad (95)$$

The limiting value of (93) as $h \rightarrow 0$ is also of practical interest since it represents a spherical shape of radius a_1 . By using L'Hospital's rule it is possible to remove the indeterminacies occurring in (93), (94) and (95) in the limit $h \rightarrow 0$ which result in the following expressions for propeller induced vibratory force given in (89).

$$\tilde{T}_1 = \pi \tilde{\delta} \rho_F \left(\frac{a_1}{x_p} \right)^3 \left(4V^2 + b^2 \omega^2 x_p^2 \right)^{\frac{1}{2}} e^{i(b\omega t + \theta)} \quad (96)$$

where the phase angle θ is given by

$$\theta = \operatorname{tg}^{-1} \left(- \frac{b\omega x_p}{2V} \right) \quad (97)$$

In a similar manner the wake fraction for a spherical shape may be obtained directly from (87) when the limit $h \rightarrow 0$ is applied. The resulting expression is

$$\bar{w}_e = \left(\frac{a_1}{x_p} \right)^3 \left[1 - \left(\frac{a_1}{x_p} \right)^3 \right] \left\{ 1 + \frac{A}{4\pi x_p^2} \left[\sqrt{1 + \bar{C}_{T+1}} - 1 \right] \right\} \quad (98)$$

which can be also obtained by applying the Lagally theorem to a sphere represented by an x-oriented doublet at the origin.

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